Stock Price Co-Movement and the Foundations of Pairs Trading

Adam Farago and Erik Hjalmarsson*

*Both authors are at the Department of Economics and the Centre for Finance, University of Gothenburg. Contact information: P.O. Box 640, SE 405 30 Gothenburg, Sweden. Email: adam.farago@cff.gu.se and erik.hjalmarsson@economics.gu.se. We have benfitted from comments by Kim Christensen, David Edgerton, Randi Hjalmarsson, Richard Payne, Joakim Westerlund, Pär Österholm, as well as by seminar participants at CREATES, Lund University, Örebro University, the Southampton Finance and Econometrics Workshop 2016, and the European Summer Meeting of the Econometric Society, Lisbon 2017. The authors gratefully acknowledge financial support from the Nasdaq Nordic Foundation.

Abstract

We study the theoretical implications of cointegrated stock prices on the profitability of pairs trading strategies. If stock returns are fairly weakly correlated across time, cointegration implies very high Sharpe ratios. To the extent that the theoretical Sharpe ratios are "too large," this suggests that either (i) cointegration does not exist pairwise among stocks, and pairs trading profits are a result of a weaker or less stable dependency structure among stock pairs, or (ii) the serial correlation in stock returns stretches over considerably longer horizons than is usually assumed. Empirically, there is little evidence of cointegration, favoring the first explanation.

I. Introduction

Pairs trading is an investment strategy based on the notion of two stock prices "co-moving" with each other. If the two prices diverge, a long-short position can be used to profit from the expected future re-convergence of the prices. Although the pairs can be formed on fundamental similarities between firms, the modern incarnation of the strategy is typically based on statistical principles, picking pairs of stocks whose share prices have previously moved closely together according to some statistical measure. In a seminal study, Gatev, Goetzmann and Rouwenhorst (2006, GGR henceforth) documented strong and consistent excess returns for a simple statistical pairs trading strategy, applied to the CRSP universe of U.S. stocks.[1] In econometric terms, the pairwise price patterns that give rise to pairs trading profits are consistent with the existence of cointegration among stock prices, and the notion of price cointegration is often used to motivate why pairs trading might be profitable (e.g.,

[1]Profitability of pairs-trading strategies has also been documented for other stock markets. For instance, Bowen and Hutchinson (2016) analyze pairs trading on the U.K. equity market and find results similar to those of GGR. Jacobs and Weber (2015) analyze individual stock data from 34 international markets and find that pairs-trading profits appear to be a consistent feature across these markets. These studies also show that pairs-trading returns do not seem to be explained by traditional factors such as market, size, value, momentum, and reversals. Do and Faff (2010) verify that pairs trading profits persist in U.S. samples dating after those used in GGR.

GGR, De Rossi, Jones, Lancetti, Jessop and Sefton (2010), and Ardia, Gatarek, Hoogerheide and van Dijk (2016)).

The purpose of the current paper is to evaluate whether cointegration among stock prices is indeed a realistic assumption upon which to justify pairs trading. In particular, we derive the expected returns and Sharpe ratios of a simple pairs trading strategy, under the assumption of pairwise cointegrated stock prices, allowing for a flexible specification of the stochastic process that governs the individual asset prices. Our analysis shows that, under the typical assumption that stock returns only have weak and fairly short-lived serial correlations, cointegration of asset prices would result in extremely profitable pairs trading strategies. In a cointegrated setting, a typical pairs trade might easily have an annualized Sharpe ratio greater than ten, for a single pair, ignoring any diversification benefits of trading many pairs simultaneously. Cointegration of stock prices therefore appears to deliver pairs trading profits that are "too good to be true."

The existence of cointegration essentially implies that the deviations between two non-stationary series is stationary.[2] The speed at which the two series converge back towards each other after a given deviation depends on the short-run, or transient, dynamics in the two processes. If there are relatively long-lived transient shocks to the series, the two processes might diverge from each other over long periods, although cointegration ensures that they eventually converge. If the transient dynamics are short-lived, the two series must converge very quickly, once they deviate from each other. In the latter case, most shocks to the series are of a permanent nature and therefore subject to the cointegrating restriction, which essentially says that any permanent shock must affect the two series in an identical manner.

To put cointegration in more economic terms, consider a simple example of two different car manufacturers. If both of their stock prices are driven solely by a single common factor, e.g., the total (expected long-run) demand for cars, then the two stock prices could easily

[2]This informal discussion implicitly assumes that the two process are cointegrated with cointegration vector $(1, -1)$, but clearly the same basic intuition holds with a vector $(1, -\gamma)$, where the second series is multiplied by γ.

be cointegrated. However, it is more likely that the stock prices depend on firm-specific demands, which contain not only a common component but also idiosyncratic components. In this case, the idiosyncratic components of demands will cause deviations between the two stock prices, and price cointegration would require that the idiosyncratic demands only cause temporary changes in the stock prices. That is, cointegration imposes the strong restriction that any idiosyncratic effects must be of a transient nature, such that they do not cause a permanent deviation between the stock prices of different firms.

In the stock price setting considered here, most price shocks are usually thought to be of a permanent nature. For instance, under the classical random walk hypothesis, *all* price shocks are permanent. Although current empirical knowledge suggests that there are some transient dynamics in asset prices, these are usually thought to be small and short lived. In this case, if two stock prices are cointegrated, there is very little scope for them to deviate from each other over long stretches of time. Thus, when a transient shock causes the two series to deviate, they will very quickly converge back to each other. Such quick convergence is, of course, a perfect setting for pairs trading, and gives rise to the outsized Sharpe ratios implied by the theoretical analysis.

The theoretical analysis thus predicts that cointegration among stock prices leads to statistical arbitrage opportunities that are simply too large to be consistent with the notion that markets are relatively efficient, and excess profits reasonably hard to achieve. Or, alternatively, the serial correlation in stock returns must be considerably longer-lived than is usually assumed, with serial dependencies stretching at least upwards of six months. However, such long-lived transient dynamics imply a rather slow convergence of prices in pairs trades, at odds with the empirical evidence from pairs trading studies (e.g. Engelberg, Gao and Jagannathan (2009), Do and Faff (2010), and Jacobs and Weber (2015)).

The tension between traditional random walk efficiency and stock price cointegration is not a new idea, as evidenced by remarks in Granger (1986). Granger's work was followed by many empirical studies of cointegration among stock prices, particularly for groups of international stock price indexes (e.g., Kasa (1992) and Corhay, Tourani Rad and Urbain (1993)). Richards (1995) provides a nice summary of this earlier literature, and argues that there is no empirical evidence of cointegration among international stock indexes, once appropriate econometric inference is conducted. Our current study contributes to this previous literature by explicitly quantifying the "profit opportunities" implied by pair-wise cointegration of asset prices.

In the second part of the paper, we evaluate to what extent there is any support in the data for the predictions of the cointegrated model. Our main empirical goal is to determine whether cointegration of stock prices is likely to exist for pairs of stocks where each of the two stocks in the pair is issued by a different firm. We refer to such pairs formed by stocks of two different firms as *ordinary pairs* throughout the paper. The analysis consists of two parts. First, we calculate empirical Sharpe ratios from the implementation of a pairs trading strategy similar to that analyzed in GGR. Second, we quantify to what extent the estimated model parameters are at all close to satisfying the restrictions implied by cointegration.

The empirical analysis is based on stocks traded on the Stockholm stock exchange. A relatively unique feature of the Swedish stock market, namely the wide-spread use of listed A- and B- shares, gives rise to a very useful control group of stock pairs. A- and B-shares of a given company are traded openly on the same exchange, provide identical ownership fractions, and are claims to the exact same cash flow. The only difference between them is that A-shares give the holder more votes than the B-shares. Since A- and B-shares of a given firm are claims to the exact same dividends, their prices are likely to be highly correlated. In fact, as shown by Bossaerts (1988), one would expect the two prices to be cointegrated.[3] The A-B pairs can therefore be seen as a form of control group, for which we would expect

[3]Bossaerts (1988) derives a general equilibrium model with cointegrated asset prices. However, the key assumption in his model is that the dividend processes are, in fact, themselves cointegrated, which in turn implies cointegration among the price processes. The result is therefore not particularly surprising, since cointegration among dividends effectively implies that, in the long-run, certain asset combinations will be claims to the same cash flows. That is, from a long-run perspective, the cointegrated assets are essentially cash-flow equivalent.

cointegration to hold.[4] If we find that the restrictions implied by cointegration are (much) further away from being satisfied for the ordinary pairs than they are for the control group of A-B stock pairs, we view this as reasonably convincing evidence that cointegration among ordinary pairs is unlikely. Again, we would like to emphasize that our main empirical question is whether cointegration of prices is likely to exist among *ordinary* (non A-B) pairs of stocks. The results for the A-B pairs should be viewed as a form of calibration of the empirical methods, providing a reasonable set of benchmark estimates against which we can compare the results for the ordinary pairs.

There are two main findings from the empirical analysis. First, before-cost Sharpe ratios from trading A-B pairs are mostly in line with the predictions of the cointegrated model, and they are considerably higher than those that can be attained when trading ordinary pairs.[5] Second, the restrictions implied by cointegration are far from being satisfied for all the possible ordinary pairs, and the parameter estimates are uniformly closer to satisfying the cointegrating restrictions for all the A-B pairs than for any of the ordinary stock pairs.

The theoretical and empirical analysis together strongly suggest that cointegration is *not* a likely explanation for the profitability of pairs trading strategies using ordinary pairs of stocks. Pairs trading is based on the idea of stock prices co-moving with each other, and that deviations from this co-movement will be adjusted and reverted, such that prices eventually converge after deviating. Profitability of such strategies is consistent with cointegration, but cointegration is not a necessary condition for pairs trading to work. Instead, it is quite likely that pairs trading profits arise because over shorter time spans, asset prices on occasion move together. This could, for instance, be due to fundamental reasons, such as a common

[4] Pairs trading of A-B pairs likely occurs, and is fully consistent with a setting where the pairs-traders act as arbitrageurs that enforce the arbitrage relationship between the stocks, as suggested by the model of Bossaerts (1988). Such trading need not lead to outsized profits because the A and B prices track each other very closely and the scope for making large monetary returns are likely limited. Our empirical results are consistent with this claim and in Section III.D we derive theoretical results that explain how this behaviour of A-B prices relate to the main theoretical results presented in this study.

[5] We also provide a detailed discussion on the effect of transaction costs, both in the theoretical and the empirical parts of the paper. In particular, we show that transactions costs tend to mostly eliminate the returns from A-B pairs trading.

and dominant shock affecting all stocks in a given industry. This view is supported by the findings in Engelberg et al. (2009) and Jacobs and Weber (2015) who document that (quick) convergence of pairs is more likely when the divergence is caused by macroeconomic news, rather than firm specific news. One could, of course, always claim that such stories are consistent with stocks *occasionally* being cointegrated, but since cointegration is defined as a long-run property such statements make little sense.

In conclusion, cointegration of stock prices, for pairs of stocks with claims to different cash flows, is unlikely for the simple reason that it would provide unrealistically large statistical arbitrage opportunities. The analysis highlights the strength of a cointegrating relationship in a setting where there are very weak short-run dynamics, and essentially shows that one cannot expect cointegration of stock prices unless there is a "mechanical" relationship that links the two assets together, as in the A-B share case discussed above.

The remainder of the paper is a follows. Section II sets up a model of cointegrated stock prices and Section III derives the main theoretical predictions for pairs trading returns. The empirical analysis is conducted in Section IV, and Section V concludes. Technical proofs and some supplemental material are found in the Appendix.

II. A Model of Cointegrated Stock Prices

We start with formulating a very general time-series model for stock returns. We assume that the returns on a given pair of stocks follow a bivariate Vector Moving Average (VMA) process, with a possibly infinite lag length. Such a process is often referred to as a linear process. It follows from the Wold decomposition (e.g., Wold (1938) and Brockwell and Davis (1991)) that any well-behaved covariance stationary process can be represented as a (vector) moving average process. Imposing a VMA structure is therefore a very weak assumption. At the same time, as illustrated in detail below, this representation allows for a very simple and clear analysis of cointegration in the corresponding price processes. In the interest of generality, the model is formulated for a k-dimensional vector of cointegrated prices, with $k = 2$ corresponding to the standard pairs trading setting.

A. A VMA Representation of Stock Returns and Stock Prices

Let y_t be a $k \times 1$ vector of (log-) stock prices, and let the first difference of y_t, $\Delta y_t = y_t - y_{t-1}$, represent the corresponding vector of (log-) returns. The returns are assumed to satisfy

$$\Delta y_t = \mu + u_t, \tag{1}$$

where μ is a constant vector and u_t is a stochastic process that follows an infinite VMA process,

$$u_t = C(L)\,\epsilon_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}, \tag{2}$$

with $\epsilon_t \equiv iid(0, \Sigma)$ and Σ a positive definite covariance matrix. u_t and ϵ_t are $k \times 1$ vector processes and c_j, $j = 0, 1, 2, ...$, are $k \times k$ coefficient matrices. $C(L) = \sum_{j=0}^{\infty} c_j L^j$, where L is the lag-operator, and $C(1) = \sum_{j=0}^{\infty} c_j$. In order to justify the BN-decomposition used below, the sum in $C(1)$ needs to converge sufficiently fast. A sufficient condition is given by $\sum_{j=0}^{\infty} j\,||c_j|| < \infty$ (Phillips and Solo (1992)). In order to avoid degenerate cases, it is also assumed that at least one element in the $k \times k$ matrix $C(1)$ is non-zero. This specification of u_t represents a stationary $(I(0))$ mean-zero vector process with a long-run covariance matrix $\Omega = C(1)\,\Sigma C(1)'$.[6] In order to make the system identifiable, the normalization $c_0 = I$ is imposed.

[6] In the usual notation of stochastic processes, $I(1)$ denotes a process integrated of order 1 (i.e., a unit-root process) and $I(0)$ denotes a covariance stationary process (the first difference of an $I(1)$ process).

The price vector, y_t, is obtained by summing up over the returns, Δy_t,

$$y_t = y_0 + \mu t + \sum_{i=1}^{t} u_t, \tag{3}$$

where y_0 represents an initial condition. This is a VMA representation of a unit-root nonstationary $(I(1))$ process.

B. Cointegration in a VMA Process

The VMA representation allows for a very simple and intuitive analysis of cointegration. Using the BN-decomposition (Beveridge and Nelson (1981)), we can write

$$u_t = C(L)\,\epsilon_t = C(1)\,\epsilon_t + \tilde{\epsilon}_{t-1} - \tilde{\epsilon}_t, \tag{4}$$

where

$$\tilde{\epsilon}_t = \tilde{C}(L)\,\epsilon_t = \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t-j}, \tag{5}$$

and

$$\tilde{c}_j = \sum_{s=j+1}^{\infty} c_s. \tag{6}$$

The $I(1)$ price process, y_t, can therefore be written as,

$$\begin{aligned} y_t &= y_0 + \mu t + C(1) \sum_{i=1}^{t} \epsilon_i + \sum_{i=1}^{t} (\tilde{\epsilon}_{i-1} - \tilde{\epsilon}_i) \\ &= \mu t + C(1) \sum_{i=1}^{t} \epsilon_i - \tilde{\epsilon}_t + (y_0 + \tilde{\epsilon}_0), \end{aligned} \tag{7}$$

using the fact that $\sum_{i=1}^{t} (\tilde{\epsilon}_{i-1} - \tilde{\epsilon}_i) = \tilde{\epsilon}_0 - \tilde{\epsilon}_t$. The representation of the price process in equation (7) shows that the price can be written as the sum of four different components: (i) a deterministic trending component (corresponding to the equity premium), (ii) a non-stationary ($I(1)$) stochastic martingale component, (iii) a transitory ($I(0)$ stationary) "noise" component ($\tilde{\epsilon}_t$), and (iv) an initial conditions component.

Cointegration of a vector $I(1)$ process implies that there exists a linear combination, $\beta' y_t$, which is $I(0)$ stationary for some $\beta \neq 0$. The $I(1)$ component in equation (7) is given by the martingale process, $C(1) \sum_{i=1}^{t} \epsilon_i$. If $\beta \neq 0$ is a cointegrating vector for y_t, it must hold that β eliminates the martingale component of y_t, i.e., $\beta' C(1) = 0$. Typically, it is also assumed that the deterministic trend is eliminated through cointegration, such that $\beta' \mu = 0$, and we will maintain this assumption throughout the paper.[7] That is, if β is a cointegrating vector, it follows from equation (7) that,

$$\beta' y_t = \beta' \mu t + \beta' C(1) \sum_{i=1}^{t} \epsilon_i - \beta' \tilde{\epsilon}_t + \beta' (y_0 + \tilde{\epsilon}_0) = -\beta' \tilde{\epsilon}_t + \beta' (y_0 + \tilde{\epsilon}_0) . \tag{8}$$

The cointegrated combination of y_t is made up of a transitory ($I(0)$) stochastic component, and the initial condition. Pairs-trading strategies are based on standardized price processes (total return indexes), initiated at some pre-specified value, and with little loss of generality we therefore set $\beta' (y_0 + \tilde{\epsilon}_0) = 0$.[8]

C. Implicit Restrictions in the Cointegrated Model

The cointegrated model specified above is stated in very general terms, relying essentially only on the assumption that returns follow a linear process. In the bivariate case ($k = 2$) with cointegrating vector $\beta = (1, -1)'$, which would be the typical pairs trading setting, the cointegrating relationship leads to some implicit restrictions on the model, as outlined below.

[7] Allowing for a non-zero deterministic trending component in the cointegrated combination implies that the linear combination $\beta' y_t$ is $I(0)$ stationary around a deterministic trend, rather than around a constant. Such a specification seems quite removed from the general idea of pairs trading, and indeed seems quite unlikely to occur in any empirical situation.

[8] That is, for $\beta = (1, -1)$ this implies that the two standardized price processes are initiated at the same value. Imposing $\beta' (y_0 + \tilde{\epsilon}_0) = 0$ has little impact on the derivations, but without this restriction one would need to explicitly subtract the initial state from the current one in certain expressions.

Later, we will use these restrictions to empirically evaluate the presence of cointegration.

First, there are restrictions on the VMA coefficients. Denote the moving average coefficient matrices, for each lag j, as

$$c_j = \begin{bmatrix} \psi_{11,j} & \psi_{12,j} \\ \psi_{21,j} & \psi_{22,j} \end{bmatrix}, \tag{9}$$

with $c_0 = I$. Define $\psi_{kl} = \sum_{j=1}^{\infty} \psi_{kl,j}$, for $k, l = 1, 2$, and it follows that

$$C(1) \equiv \sum_{j=0}^{\infty} c_j \equiv I + \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} = \begin{bmatrix} 1+\psi_{11} & \psi_{12} \\ \psi_{21} & 1+\psi_{22} \end{bmatrix}. \tag{10}$$

If $\beta = (1, -1)'$ is a cointegrating vector, then $\beta' C(1) = 0$ implies

$$\psi_{21} = 1 + \psi_{11} \qquad \text{and} \qquad \psi_{12} = 1 + \psi_{22}. \tag{11}$$

Second, there are restrictions on the long-run covariance matrix of returns. Let $\Gamma_j \equiv E\left[(\Delta y_t - \mu)\left(\Delta y'_{t+j} - \mu\right)\right]$ denote the jth autocovariance of the returns Δy_t. The long-run covariance matrix of Δy_t is then defined as

$$\Omega = \sum_{j=-\infty}^{\infty} \Gamma_j = \Gamma_0 + \sum_{j=1}^{\infty} \left(\Gamma_j + \Gamma'_j\right). \tag{12}$$

In the VMA model, $\Omega = C(1)\Sigma C(1)'$, and under cointegration,

$$\beta'\Omega = \beta'\left(C(1)\Sigma C(1)'\right) = \left(\beta' C(1)\right)\Sigma C(1)' = 0, \tag{13}$$

where the last equality follows from $\beta' C(1) = 0$. If $\beta = (1, -1)'$, this implies that

$$\beta'\Omega = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{21} \\ \omega_{21} & \omega_{22} \end{bmatrix} = \begin{bmatrix} \omega_{11} - \omega_{21} & \omega_{21} - \omega_{22} \end{bmatrix} = 0, \tag{14}$$

such that all the elements in the long-run covariance matrix must be identical in this case.

III. Return Properties of a Pairs Trading Strategy

A pairs trading strategy for a given pair of stocks is usually defined along the following lines. If the difference between the (standardized) prices of stock 1 and stock 2 exceed a given threshold, a short position is taken in the stock with the relatively higher price and a long position in the stock with the relatively lower price. The long and short positions are of identical magnitude, resulting in a zero cost strategy. The threshold is defined in terms of the unconditional standard deviation of the observed difference between the two price processes. A two standard deviation difference is a standard trigger of a pairs trade. The position is closed either after a given amount of time, or after the two prices converge. In the theoretical analysis below, we restrict ourselves to fixed holding periods, such that the position always closes after a given number of days. The joint price process used for measuring divergence is defined as the total return indexes for the two stocks, initiated at some prior date.

These conditions extend naturally to the formal setting considered here, with y_t interpreted as a total return series for the stocks; for simplicity, we continue referring to y_t as the price process. If y_t is a bivariate price process with cointegrating vector $\beta = (1, -1)'$, the *change* in $\beta' y_t = y_{1,t} - y_{2,t}$ represents the return on a pairs trading strategy triggered by a decline in price 1 relative to price 2. If price 1 was instead higher than price 2, the pairs trade would take on the negative position, $-\beta$. In the analysis below, without loss of generality, we define a pairs trade as taking on a position β, with the implicit understanding that if the price spread is reversed, the opposite position would be used. More generally, for a k-dimensional price process y_t, with cointegration vector β, the change in $\beta' y_t$ represents the return on a generalized pairs trading strategy, involving k different stocks. Such strategies represent a natural extension of the pairs trading idea, as pointed out by GGR, and the main results below are derived for a general k-dimensional price process with arbitrary cointegration vector β. However, we focus the discussion on the standard bivariate case with $\beta = (1, -1)'$.

A. The Finite Lag Case

If we assume that Δy_t follows a finite order VMA process, such that $u_t = C(L)\epsilon_t = \sum_{j=0}^{q} c_j \epsilon_{t-j}$, with $q < \infty$, explicit results can be calculated for the returns on the pairs trading strategy where the holding period p is identical to the lag length q. In particular, Theorem 1 below derives explicit expressions for the conditional moments of a pairs trading strategy in this case.

Theorem 1 *Suppose $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ is a $k \times 1$ dimensional returns process, with $\epsilon_t \equiv iid(0, \Sigma)$, $C(L) = \sum_{j=0}^{q} c_j L^j$, $q < \infty$, and $C(1) \neq 0$. The corresponding price process is given by y_t and the $q-$period returns on the pairs trading strategy is defined as $r_{t \to t+q} \equiv \sum_{j=1}^{q} \Delta\beta' y_{t+j} = \beta' y_{t+q} - \beta' y_t$. If y_t is cointegrated with cointegration vector β, the following results hold for the returns on the pairs trading strategy.*

i. *The time t conditional expected $q-$period return is given by*

$$E_t[r_{t \to t+q}] = -\beta' y_t. \tag{15}$$

ii. *The time t conditional variance of the $q-$period return is given by*

$$Var_t(r_{t \to t+q}) = Var(\beta' y_t), \tag{16}$$

where $Var(\beta' y_t)$ is the unconditional variance of $\beta' y_t$.

iii. *The time t conditional Sharpe ratio for the $q-$period return is given by*

$$SR_t\left(r_{t\to t+q}\right) \equiv \frac{E_t\left[r_{t\to t+q}\right]}{\sqrt{Var_t\left(r_{t\to t+q}\right)}} = \frac{-\beta' y_t}{\sqrt{Var\left(\beta' y_t\right)}}. \tag{17}$$

The results highlight several important points.

1. In the bivariate pairs trading case with $\beta = (1, -1)'$, the conditional expected $q-$period returns are exactly proportional to the deviation between the two prices, $E_t\left[r_{t\to t+q}\right] = -\beta' y_t = y_{2,t} - y_{1,t}$. That is, the larger the deviation between the prices, the greater the expected returns. Further, the conditional variance of the $q-$period pairs trading returns is identical to the *unconditional* variance of the spread between the two price processes.

2. The VMA parameters, which govern the dynamics of the price processes, do not explicitly enter into the expected returns and variance formulas. In essence, the cointegrating relationship, along with the lag length in the model, pins down the speed of convergence over the next q periods in a cointegrated vector moving average model with q lags.[9]

3. Suppose that we observe a negative two standard deviation outcome of the spread, $\beta' y_t$. That is, suppose $-\beta' y_t = 2\sqrt{Var\left(\beta' y_t\right)}$. In this case,

$$SR_t\left(r_{t\to t+q}\right) = \frac{E_t\left[r_{t\to t+q}\right]}{\sqrt{Var_t\left(r_{t\to t+q}\right)}} = \frac{-\beta' y_t}{\sqrt{Var\left(\beta' y_t\right)}} = \frac{2\sqrt{Var\left(\beta' y_t\right)}}{\sqrt{Var\left(\beta' y_t\right)}} = 2. \tag{18}$$

   If q is measured in days and there are 250 trading days during the year, the annualized Sharpe ratio is

$$SR_{ann} = \sqrt{\frac{250}{q}} \times 2. \tag{19}$$

[9] If one were to calculate the expected returns over other periods than the $q-$period horizon used in Theorem 1, the answer would generally depend on the lag coefficients explicitly, as seen in Theorem A1 in the Appendix.

Table 1: Properties of q-Period Pairs Trading Returns

The table presents annualized Sharpe ratios of pairs trading strategies where a trade is initiated by a two standard deviation price spread and is held open for q periods, and the returns are generated by a VMA(q) model. The formula for the annualized Sharpe ratios is given in equation (19).

q	1	5	10	25	50	125	250
SR_{ann}	32	14	10	6.3	4.5	2.8	2

Table 1 reports annualized Sharpe ratios for different values of q. For instance, if returns follow a VMA(10) process where the corresponding price processes are cointegrated, and one puts on pairs trades with a ten-day holding period when the spread is two standard deviations wide, the strategy has an annualized Sharpe ratio of 10.

4. The results hold for general $k-$dimensional cointegrated price processes, with cointegration vector β. Conditional on a given value of $\beta' y_t$, the expected returns and Sharpe ratios are unaffected by the dimension of the system (i.e., by the value of k).

The results in Theorem 1 provide a very clear picture of the return properties of a pairs trading strategy when the returns follow a VMA of some finite order q, and the holding period for the trading strategy is equal to q periods (days). For small to moderate values of q, such holding periods are quite sensible and realistic. However, as q increases, and in particular as $q \to \infty$, it is no longer feasible to consider holding periods that are equal to q days. Instead, we want to consider fixed holding periods, as well as allowing infinite values for the lag length q.

B. Fixed Holding Periods and the Infinite Lag Case

We start with deriving theoretical results for a holding period $p = 1$, allowing for lag length $q = \infty$. As shown in Theorem A1 in Appendix C, for an arbitrary value of the lag length q (including $q = \infty$), the conditional expected pairs trading return from t to $t + 1$ is not solely a function of the distance between the two price processes, $\beta' y_t$, but also depends explicitly on the realizations of the previous shocks, ϵ_{t-j}, and the MA coefficients, c_j. The simple mapping between the price difference, $\beta' y_t$, and the Sharpe ratio of the pairs trading strategy, seen in Theorem 1, is therefore no longer present. That is, conditioning on the price difference is no longer sufficient to pin down the conditional Sharpe ratio for a given pairs trade.

In particular, as shown in Theorem A1, the one-period conditional Sharpe ratio is given by

$$SR_{t,t\to t+1} = \frac{\beta' \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}}{\sqrt{\beta'\Sigma\beta}}. \tag{20}$$

This expression is not directly amenable to the analysis of pairs trading strategies that are conditioned on a certain price divergence (i.e., $\beta' y_t$) between the two stocks.[10] The sequence $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ is not a directly observable quantity, and statements conditional on a specific realization of this sequence are not particularly useful. To get around this issue, we consider the notion of an *unconditional* pairs trade: at some arbitrary time t, an investor puts on a pairs trade *without* conditioning on the price difference or any other information. This is obviously not an attractive strategy, with an expected return equal to zero.[11] However, it enables us to think of the sequence $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ as a random, rather than a realized quantity. Formally, given information at time t, $\sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}$ takes on a fixed (non-stochastic) value, which in turn delivers a fixed conditional Sharpe ratio. If one does not condition on information formally realized at time t, $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ is a random sequence and $SR_{t\to t+1}$ a random variable. In particular, we can think of $SR_{t\to t+1}$ as the time t stochastic Sharpe ratio facing the investor who puts on the unconditional pairs trade at time t. Had the investor observed $\{\epsilon_{t-j}\}_{j=0}^{\infty}$, the Sharpe ratio would have been a fixed number, but without this information, it is a random variable. Under more specific assumptions on the sequences c_j and ϵ_t, an explicit distribution can be derived for the stochastic Sharpe ratio:

[10] As made clear in equation (A27) in the Appendix, the conditional Sharpe ratio partly depends on the price difference $\beta' y_t$, but also on the sequence of previous shocks.

[11] To be clear, we define the pairs trade as always going long stock 1 and short stock 2, such that the position is given by $\beta = (1, -1)$. Thus, since the investor does not condition at all on the current prices, he is just as likely to put on a trade in the "wrong" direction (i.e., go long stock 1 when it has increased in price relative to stock 2) as in the right direction. As seen in Theorem 2, the expected return is indeed equal to zero.

Theorem 2 *Suppose* $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ *is a bivariate returns process, with* $\epsilon_t \equiv iidN(0, \Sigma)$, $C(L) = \sum_{j=0}^{\infty} c_j L^j$, *and* $C(1) \neq 0$. *The corresponding price process is given by* y_t, *and the returns on the pairs trading strategy is defined as* $r_{t \to t+1} \equiv \Delta\beta' y_{t+1}$. *Further, assume that the coefficients* c_j *can be written as*

$$c_j = h(j) \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{21}$$

where $h(j)$ *is a convergent series such that* $H_\infty \equiv \sum_{j=1}^{\infty} h(j) < \infty$. *In addition, define* $H_\infty^{(2)} \equiv \sum_{j=1}^{\infty} h(j)^2$. *If* y_t *is cointegrated with cointegration vector* $\beta = (1, -1)$, *the one-period Sharpe ratio for the unconditional pairs trading strategy is distributed according to*

$$SR_{t \to t+1} \sim N\left(0, \frac{H_\infty^{(2)}}{H_\infty^2}\right). \tag{22}$$

Theorem 2 provides the distribution of the Sharpe ratio under the assumption of normally distributed innovations, and MA coefficients that are proportional to some function $h(j)$. The series $h(j)$ is assumed to be convergent. If $q < \infty$, this restriction is trivially satisfied and the same results hold, with H_∞ replaced by $H_q \equiv \sum_{j=1}^{q} h(j)$ and $H_\infty^{(2)}$ by $H_q^{(2)} \equiv \sum_{j=1}^{q} h(j)^2$. As is apparent from the definitions of H_∞^2 and $H_\infty^{(2)}$, the distribution of the Sharpe ratios is invariant to the overall scale of the lag coefficients, (i.e., the values of a, b, c, d in equation (21)), and only depends on the relative weights attributed to each lag (i.e., the shape of the function $h(j)$). This result echoes that in Theorem 1, where the Sharpe ratio is completely invariant to the lag coefficients c_j.

How can one link the distribution of the stochastic Sharpe ratio in equation (22) with the actual fixed conditional Sharpe ratio from a conditional pairs trading strategy? Suppose the conditional pairs trading Sharpe ratio is monotonically increasing in the observed price deviation. I.e., the larger $\beta' y_t$ is, the greater is the Sharpe ratio. In that case, the two standard deviation outcome of the Sharpe ratio distribution should correspond to conditional pairs trades triggered by a two standard deviation price divergence. That is, from the distribution of the Sharpe ratios for the *unconditional* pairs trading strategy, we can infer the Sharpe ratios of the *conditional* pairs trading strategy.

From Theorem A1 in Appendix C, it appears that the Sharpe ratio is increasing in the price deviation. It is not clear that the relationship is monotone, however, since the price difference is a function of the past shocks that also appear in the Sharpe ratio expression (equation (A27)). Therefore, we cannot say for certain that a two standard deviation divergence between the price processes corresponds to a two standard deviation outcome of the Sharpe ratio. However, it would be surprising if the speed of convergence did not increase in the size of the price deviation, and simulation results reported below strongly suggest that this is indeed the case.

In Table 2, we report the one standard deviation annual Sharpe ratio, from Theorem 2, for various parameterizations of $h(j)$. The daily one standard deviation Sharpe ratio equals $\sqrt{H_\infty^{(2)}/H_\infty^2}$, and the corresponding annualized Sharpe ratio is $\sqrt{250}\sqrt{H_\infty^{(2)}/H_\infty^2}$. We also report the Sharpe ratios from simulated pairs trades triggered at either a one or two standard deviation threshold (SR^1_{ann} and SR^2_{ann}, respectively). The details of the simulation procedure are described in Appendix A. As seen in Table 2, the simulated one standard deviation Sharpe ratios (SR^1_{ann}) are very close to the corresponding one standard deviation Sharpe ratios from the theoretical analysis $\left(\sqrt{250}\sqrt{H_q^{(2)}/H_q^2}\right)$, and the simulated Sharpe ratios appear to grow linearly with the observed price difference, measured in standard deviations. Table 2 thus gives strong support to the conjecture that the Sharpe ratio is monotonically increasing in the price difference.

Table 2 reports Sharpe ratios for various specifications of $h(j)$ in the form

$$h(j) = \frac{1}{j^\gamma} \quad \text{or} \quad h(j) = \frac{(-1)^j}{j^\gamma}. \tag{23}$$

In all cases, the MA coefficients decline in absolute magnitude according to a power function, or remain constant ($\gamma = 0$). Since most of these specifications do not result in finite $H_\infty =$

Table 2: Annualized Sharpe Ratios from One-Period Pairs Trading

The table presents annualized Sharpe ratios of pairs trading strategies where a trade is held open for one period, and the returns are generated by a VMA(q) model. The $h(j)$ specifications describe the lag structure of the VMA coefficients (see equation (21)), while $H_q \equiv \sum_{j=1}^{q} h(j)$ and $H_q^{(2)} \equiv \sum_{j=1}^{q} h(j)^2$. The SR_{ann}^1 and SR_{ann}^2 values correspond to the one- and two standard deviation strategies, respectively, and are calculated using simulated pairs trades. The columns labeled $\sqrt{250}\sqrt{H_\infty^{(2)}/H_\infty^2}$ indicate the theoretical one standard deviation Sharpe ratios.

	$q=\infty$	$q=10$			$q=250$		
$h(j)$	$\sqrt{250}\sqrt{\frac{H_\infty^{(2)}}{H_\infty^2}}$	$\sqrt{250}\sqrt{\frac{H_{10}^{(2)}}{H_{10}^2}}$	SR_{ann}^1	SR_{ann}^2	$\sqrt{250}\sqrt{\frac{H_{250}^{(2)}}{H_{250}^2}}$	SR_{ann}^1	SR_{ann}^2
$1/j^2$	10.00	10.61	10.45	20.92	10.02	9.51	19.00
$1/j$		6.72	6.58	13.21	3.32	2.25	4.51
$1/j^{0.5}$		5.39	5.22	10.45	1.29	1.18	2.33
1		5.00	4.41	8.76	1.00	0.88	1.74
$(-1)^j/j^2$	20.00	20.11	20.03	40.09	20.00	19.94	39.90
$(-1)^j/j$	29.26	30.49	28.29	56.47	29.31	28.63	57.31

$\sum_{j=1}^{\infty} h(j)$, results for finite q processes are also shown, setting $q = 10$ or 250. In the convergent cases, restricting the MA process to only 10 lags leaves results almost identical to those in the MA(∞) case, for a given specification of $h(j)$. Most specifications in Table 2 result in very high annual Sharpe ratios. This is particularly true for the alternating series, $h(j) = (-1)^j \big/ j^\gamma$, with Sharpe ratios of 40 and above for a two standard deviation strategy. It is also clear that the Sharpe ratios become smaller as $h(j)$ declines slower. This makes intuitive sense, since a more slowly declining $h(j)$ is associated with slower mean reversion in the model, or put differently, more long-lasting transient dynamics.

To some extent, the purpose of Table 2 is to evaluate whether there exists any "reasonable" parameterizations of c_j, which admit cointegration but does not result in Sharpe ratios that are too high. Note that Sharpe ratios around 2 are generally in line with those documented empirically in GGR for typical pairs trading strategies, although for an individual pair, a Sharpe ratio of 2 is probably still on the high side. The only parameterizations in Table 2 that result in annualized Sharpe ratios around 2 for a two standard deviation strategy are $h(j) = 1/j^{0.5}$ and $h(j) = 1$, with a lag length $q = 250$. More quickly declining weights (a larger γ or a smaller maximal lag length q) result in Sharpe ratios that are considerably larger. Thus, judging only by the Sharpe ratios, it would seem that a lag structure spanning upwards of a year ($q = 250$) with MA coefficients that decline no faster than a rate $1/j^{0.5}$

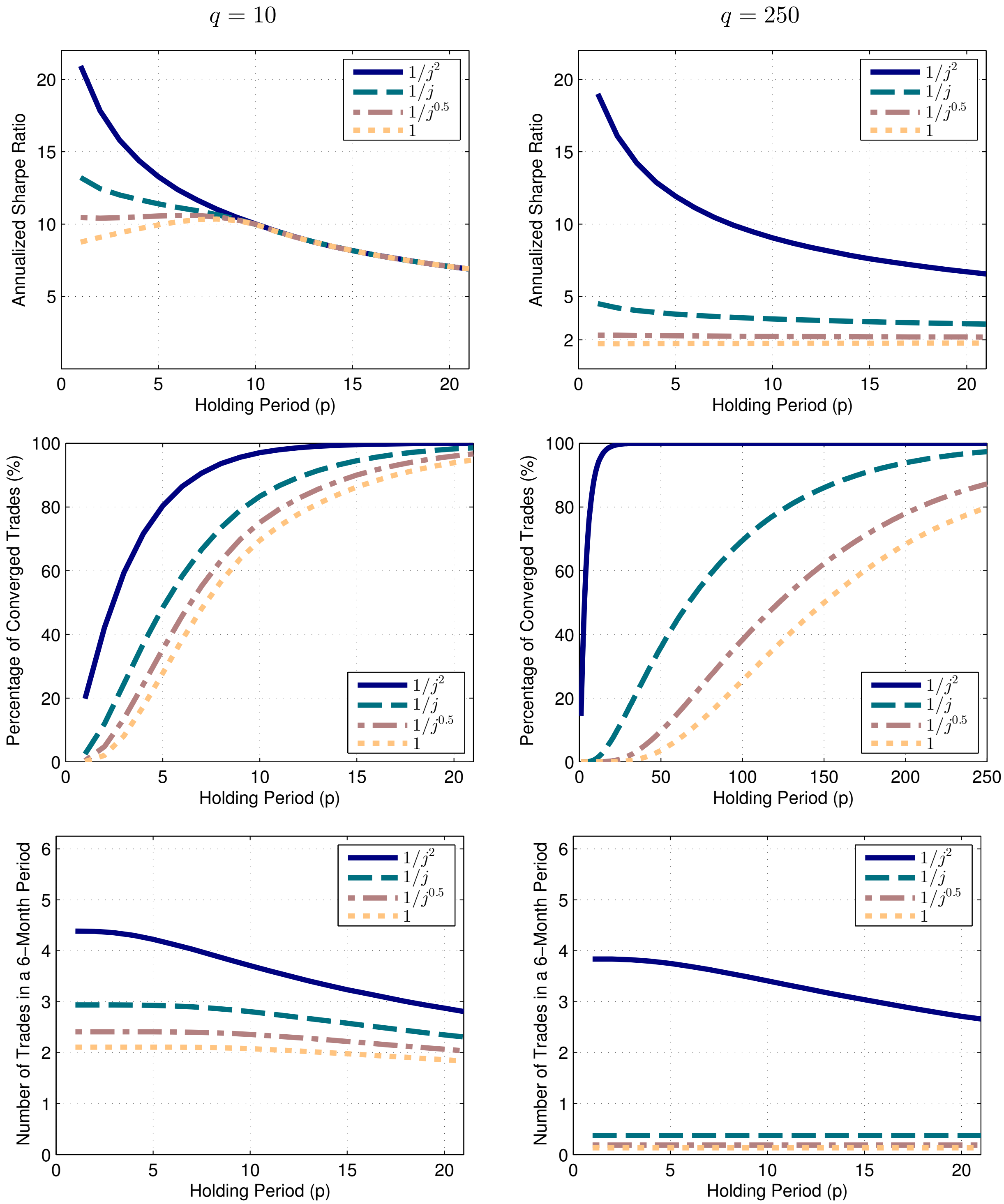
would be necessary to keep the Sharpe ratios within reasonable bounds.

Theorem 2 provides theoretical results for the case when the holding period is $p = 1$. We provide simulation results for strategies when the pairs trading position is closed after $p > 1$ periods (days); the details of the simulation design are described in Appendix A. The two top graphs in Figure 1 show annualized Sharpe ratios generated by a two standard deviation strategy for holding periods up to a month ($p = 21$). Several different VMA parameterizations are presented ($q = \{10, 250\}$ and different $h(j)$ specifications). For $p = 1$, the Sharpe ratios exactly correspond to the SR^2_{ann} values from Table 2. Focusing first on the $q = 10$ case, we see that the Sharpe ratios decline with increasing holding period when $\gamma = 2$ or 1. On the contrary, when $h(j)$ declines slowly ($\gamma = 0.5$ or 0), Sharpe ratios initially increase with the holding period. In these latter cases, since the convergence of the two prices is slower, holding the position for a few days leads to higher risk adjusted returns. When $p = q = 10$, Theorem 1 applies and the Sharpe ratio is the same for all $h(j)$ specifications. All in all, with $q = 10$, the two standard deviation strategy produces annual Sharpe ratios well above 5 for any $h(j)$ specification and holding periods at least up to a month. Turning to the $q = 250$ case, we showed in Table 2 that the parameterizations with $h(j) = 1/j^{0.5}$ and $h(j) = 1$ lead to Sharpe ratios around 2 for a one-day holding period. The top right graph in Figure 1 shows that the Sharpe ratios do not change much with the holding period in these cases, but stay around 2. Actually, we know from Theorem 1 that the Sharpe ratio exactly reaches 2 when the holding period is $p = q = 250$ days. To summarize, considering longer holding periods ($p > 1$) does not change our conclusions from Theorem 2. A lag structure spanning upwards of a year ($q = 250$), with MA coefficients that decline no faster than a rate $1/j^{0.5}$, would be necessary to keep the Sharpe ratios within reasonable bounds.

The middle two graphs in Figure 1 show the percentage of converged trades (where the two prices have converged) by period p in our simulation. We focus on the cases when $q = 250$ (middle right graph in Figure 1) and $h(j)$ declines slowly ($h(j) = 1/j^{0.5}$ or $h(j) = 1$), as these parameterizations provide empirically reasonable Sharpe ratios around 2. The convergence

Figure 1: Properties of a Two Standard Deviation Strategy for Different Holding Periods

The graphs present trade characteristics of pairs trading strategies where a trade is initiated by a two standard deviation price spread and held open for p periods (on the horizontal axis), and the returns are generated by a VMA(q) model ($q = 10$ and $q = 250$ for the graphs on the left and right, respectively). The lines correspond to different $h(j)$ specifications (see the legend). The top graphs present annualized Sharpe ratios, the middle graphs present the percentage of converged trades within p periods, and the bottom graphs present the number of opened trades within 125 trading days (6 months). All characteristics are calculated using the simulation procedure described in Appendix A.



of pairs trades in these cases is quite slow (note that the scale for p is different in this graph, compared to all the other graphs in Figure 1). Up to a holding period of one month (p up to 21), less than 1% of the trades converge. Even if we consider holding periods up to two months (p up to around 40), only around 5% of the trades converge. This is at odds with the empirical evidence from pairs trading studies, which suggest that pairs trading is a relatively fast strategy, with convergence of pairs often occurring within a month or so (e.g. Engelberg et al. (2009), Do and Faff (2010), and Jacobs and Weber (2015)).

The final two graphs at the bottom of Figure 1 further illustrate that large values of q lead to a severe slow-down of pairs trading. The graphs show how often a new trade is opened in a given pair.[12] Specifically, the bottom panels in Figure 1 show the average number of trades per 6-month period (125 trading days); the 6-month period is chosen to align with the summary statistics presented in the empirical section of the paper. For $q = 10$, this number is typically around 2.5, depending somewhat on the holding period p and the shape of the function $h(j)$. That is, on average, a new pairs trade is put on about every 50 trading days. For $q = 250$, the trading frequency is much smaller – unless the MA coefficients decline very quickly at a rate $1/j^2$ – with only about 0.25 trades in a given 6-month period or, equivalently, a new trade roughly every 500 days. To put these numbers in an empirical context, GGR find that a typical pair trades approximately 2 times in a 6-month period. This is similar to the trade frequency obtained for $q = 10$, but much more frequent than what is observed for $q = 250$ (unless $h(j) = 1/j^2$).

All the above results are derived under a VMA specification for the returns process. An alternative way of modeling stock returns is, of course, through a Vector Autoregression (VAR) model. It is well known that stationary VAR models can be inverted into VMA

[12] A pairs trade position is first opened when the spread between the two prices crosses the two standard deviation trigger point. If the pair converges during the holding period of p days, the pair is eligible to trade again immediately after closing; i.e., a position will be opened again if the prices diverge beyond the two standard deviation interval. If the pair does not converge during the holding period, the pair does not become available for trading again until it has converged. That is, at any given point, at most one position can be open in the pair, and the pair must converge between each new trade. This trading rule essentially follows that of GGR, apart from the fact that in GGR the positions are held until convergence, instead of a fixed period.

models, and vice versa for invertible VMA models. One would therefore expect the two modeling approaches to yield similar pairs trading implications. Appendix B derives a result similar to Theorem 1, but in a VAR setting. As is discussed in some detail in Appendix B, the implications for pairs trading in a cointegrated VAR setting are indeed very similar to those derived in the VMA case.

C. Is Cointegration among Stock Prices Plausible?

Theorems 1 and 2 explicitly quantify the properties of the returns from a pairs trading strategy in a cointegrated price system. Arguably, the most important determinant of the Sharpe ratio on the pairs trading strategy is the maximal lag-length q in the VMA process that governs the dynamics of the stock returns.[13] It is clear that the most outsized Sharpe ratios occur for small values of q. The parameter q can be viewed as the maximal lag length at which the returns exhibit any own- or cross-serial correlation. A value of $q = 250$ would suggest that the serial correlation in stock returns stretches back one year. Or, put differently, that it takes up to a year to fully incorporate news into stock prices, after these news are initially revealed.

Cointegration becomes a very powerful concept when coupled with asset prices, because the transitory component in asset prices is generally considered to be small and short-lived. This lack of short-run dynamics in stock prices puts strong bounds on the duration for which two cointegrated price processes can deviate from each other, and these bounds grow tighter as the temporal span of the lag effects (i.e., q) becomes smaller. In our view, the above theoretical analysis therefore implies either that (i) cointegration among stock prices does not exist (or is at least very unlikely), since the implied Sharpe ratios appear too large to be realistic, or (ii) that the serial correlation in stock returns stretches over considerably

[13]More generally, as seen in Theorem 2, Table 2, and Figure 1, the speed of the decline in the VMA lag-coefficients (i.e., the shape of the $h(j)$ function) is the primary determinant of the profitability of pairs trading strategies. In practice, distinguishing between an infinite order VMA model with quickly declining lag coefficients, and a finite order VMA model with some maximal lag length q, is essentially impossible. We therefore focus the discussion around the notion of a finite maximal lag length.

longer horizons than is usually assumed. The exact cut-off point for "too large a Sharpe ratio" is of course not precisely defined, but individual investment opportunities with Sharpe ratios above three or four should be few and far in between, especially when they can be implemented as easily as a pairs trading strategy, which requires nothing more complicated than the ability to short-sell a stock.[14] Such a threshold would suggest that the dynamics in stock returns play out over at least 6 months, and more likely 12 months ($q = 250$), in order for cointegration to be realistic. However, such long-lasting transient dynamics appears to be at odds with the empirical evidence that pairs trading is a relatively fast strategy, with a new trade in a given pair every 2-3 months and convergence typically occurring within a month or so. Figure 1 shows that trading is much less frequent and the convergence is considerably slower in the $q = 250$ case.

D. The Size of Price Deviations and Transaction Costs

The theoretical Sharpe ratios derived above are all invariant to the overall scale of the price processes. That is, the Sharpe ratios correspond to pairs trades triggered by two standard deviation price spreads, but the absolute size of that two standard deviation spread does not enter into the formulas for the Sharpe ratios. However, once one starts considering transaction costs, the actual scale of the price processes becomes important. If trading costs shave off a fixed amount (i.e., a number of percentage points) from each trade, as is usually assumed, the overall level of returns becomes highly important. Corollary 1 shows what the actual price spreads would be under cointegration, given similar parametrizations to before.

[14] There are examples of statistical arbitrage strategies that appear to deliver very high Sharpe ratios. For instance, both Nagel (2012) and Wahal and Conrad (2017) report annualized Sharpe ratios well above 5 for some strategies. Interestingly, both of these examples represent returns to some form of liquidity provision, which is nowadays closely connected with the ability to trade at low costs–much of modern market making is conducted by high-frequency traders who specialize in trading with minimum frictions. As seen in the discussion on transaction costs in Section III.D, cointegration is most likely to be present in the case when the two price processes co-move very closely, in which case the absolute level of returns from pairs trading is small, and therefore highly sensitive to transaction costs. Since trading costs can arguably be decreased by investments in trading infrastructure, these large Sharpe ratios might be viewed as partly a return on the investment in trading infrastructure.

Corollary 1 *Suppose* $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ *is a bivariate returns process with* $\epsilon_t \equiv iid(0, \Sigma)$*,* $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$*,* $C(L) = \sum_{j=0}^{q} c_j L^j$*, and* $C(1) \neq 0$*. Assume that the coefficients* c_j *can be written as in equation (21). If the corresponding price process,* y_t*, is cointegrated with cointegration vector* $\beta = (1, -1)$ *, then*

$$Var(\beta' y_t) = Var(y_{1,t} - y_{2,t}) = (\sigma_{11} + \sigma_{22} - 2\sigma_{12}) \sum_{j=0}^{\infty} \left[\frac{\left(\sum_{s=j+1}^{\infty} h(s) \right)^2}{\left(\sum_{s=1}^{\infty} h(s) \right)^2} \right]. \tag{24}$$

For ease of illustration, suppose that $\sigma_{11} = \sigma_{22}$, so that equation (24) can be written as

$$Var(y_{1,t} - y_{2,t}) = 2\sigma_{11}(1 - \rho_{12}) \sum_{j=0}^{\infty} \left[\frac{\left(\sum_{s=j+1}^{\infty} h(s) \right)^2}{\left(\sum_{s=1}^{\infty} h(s) \right)^2} \right], \tag{25}$$

where ρ_{12} is the correlation between the innovations to the two price processes. The formulation in (25) highlights the strong dependence between the variance of the price deviations, and the correlation between the innovations: as $\rho_{12} \uparrow 1$, $Var(y_{1,t} - y_{2,t}) \downarrow 0$. A higher correlation implies that the two price processes are hit by more similar shocks, limiting the size of the deviations between the two processes, keeping all else equal.[15]

In order to get a sense of the actual scale of the price deviations, suppose that σ_{11} and σ_{22} are both equal to 4.5 percent. This is similar to the average daily stock return variance in our data, and also in line with average daily variances for (large) U.S. stocks. Table 3 presents the two standard deviation spread that would trigger a pairs trade for different ρ_{12} values. In the setting of Theorem 1, where the holding period is equal to the lag length ($p = q$), these numbers also represent the expected returns on the pairs trade (see equation (15)). For instance, with $h(j) = 1/j$ and $q = 10$, the expected return over the 10-day holding period is

[15]The function $h(s)$ also plays a role. As seen in (24), the less relative mass the function $h(s)$ has for large s, the smaller the variance of the price difference, keeping σ_{11}, σ_{22}, and σ_{12} fixed. That is, as discussed previously, limiting the short-run dynamics implies smaller deviations between the two price processes, keeping all else constant.

Table 3: Theoretical Two Standard Deviation Spreads

The table presents two standard deviation spreads implied by the VMA(q) model. In particular, we report $2\sqrt{Var\left(y_{1,t}-y_{2,t}\right)}$ values calculated from equation (25) with $\sigma_{11}=0.045$, $\rho_{12}\in\{0.5,0.95\}$, $q\in\{10,250\}$, and various $h(j)$ specifications. All values in the table are expressed in percent.

	$q=10$		$q=250$	
$h\left(j\right)$	$\rho_{12}=0.5$	$\rho_{12}=0.95$	$\rho_{12}=0.5$	$\rho_{12}=0.95$
$1/j^2$	4.63	1.46	4.87	1.54
$1/j$	5.98	1.89	15.46	4.89
$1/j^{0.5}$	7.14	2.26	28.74	9.09
1	8.32	2.63	38.85	12.28
$\left(-1\right)^j/j^2$	4.37	1.38	4.36	1.38
$\left(-1\right)^j/j$	5.28	1.67	5.10	1.61

equal to 6.0 and 1.9 percent for correlations $\rho_{12}=0.5$ and $\rho_{12}=0.95$, respectively. As seen from the result in Theorem 1, the Sharpe ratio of the trade is not affected by the correlation, since the variance of the returns also decreases as the correlation increases. However, if one takes trading costs into account, this invariance of the Sharpe ratio no longer holds.

As we discuss further in the empirical section, it is reasonable to assume that the trading costs for a full round trip of a pairs trade is about 1.2 percentage points (120 basis points (bps)). Thus, the before-cost expected returns of 6.0 and 1.9 percent would reduce to 4.8 and 0.7 percent after costs, respectively. Since the variance of the returns are not affected by trading costs, the Sharpe ratios after costs decrease in proportion with returns, in this case by 20% and 65% for $\rho_{12}=0.5$ and $\rho_{12}=0.95$, respectively. Thus, transaction costs become increasingly important as $\rho_{12}\uparrow 1$.

Provided that the two standard deviation spread that triggers a pairs trade is not too small, transaction costs only have a limited impact on the theoretical Sharpe ratios. Or put differently, as long as the innovations to the two price processes are not too highly correlated, the theoretical results are only mildly affected by transaction costs. These findings also suggest a third interpretation of the results in this paper: if prices *are* cointegrated *and* have limited short-run dynamics, the correlation between the innovations must be near unity. In this case, although the theoretical Sharpe ratios are unaffected, the price deviations are small enough that they are likely difficult to profit substantially from, given transaction costs. As discussed in the empirical section, prices for A- and B-shares issued by the same firm fit this description quite well, whereas ordinary (non A-B) pairs show much larger spreads and never exhibit very high correlations between the innovations.

E. Misclassification of Pairs

The above theoretical framework implicitly assumes that pairs of stocks with cointegrated prices are already identified by the investor. In practice, even if cointegrated stock prices do exist, the investor would have to do some initial screening to find pairs of stocks with cointegrated prices. Such an empirical classification would naturally run the risk of misclassifying some pairs as cointegrated when they are, in-fact, not.

In the theoretical analysis, we only consider the return properties of a single pairs trade, but clearly one can combine such trades into a portfolio of pairs trades. Specifically, suppose that some fraction λ of the pairs identified for pairs trading are not cointegrated, whereas the other pairs are cointegrated. The cointegrated pairs would result in pairs-trading returns that satisfy the results derived above. However, the non-cointegrated pairs would likely perform considerably worse. For simplicity, suppose that the mean returns on the non-cointegrated pairs trades are equal to zero. In this case, assuming that the other properties of the non-cointegrated pairs are similar to those of the cointegrated pairs, the Sharpe ratio of the portfolio of pairs trades would drop by a fraction λ, compared to the case of a portfolio with no misclassified pairs.[16] If λ was equal to, say, 20 percent and the Sharpe ratio in the correctly classified pairs case was equal to 3, it would drop to 2.4. This is a substantial performance deterioration to an actual investor, but does not change the qualitative conclusions of the theoretical analysis. Therefore, unless the misclassification is of a very large order, upwards

[16]The mean-zero assumption would seem fairly conservative, essentially implying that no signal at all was identified when selecting the pair. The assumption that the Sharpe ratio is only affected through the mean returns implies that the variances and covariances are assumed identical for the cointegrated and non-cointegrated pairs. For short holding periods (e.g., 10 or 20 days), this does not seem an unreasonable assumption as the the long-short position can be scaled up or down to some target variance; in theory, the variance of the spread between two non-stationary non-cointegrated processes will increase over time, but over short horizons such effects are small. There seems to be little reason to assume that diversification benefits from non-cointegrated pairs would be worse than those from other cointegrated pairs, so the covariance assumption seems fairly innocuous.

of 50 percent or more, the potential effects on the Sharpe ratios are fairly limited in the sense that the overall message of the theoretical analysis does not get overturned in any way.

IV. Empirical Analysis

The empirical analysis is based on stocks traded on the Stockholm stock exchange. Our initial sample consists of all stocks listed in the large cap segment of the Nasdaq Stockholm exchange as of June 1, 2015 and the sample period is from January 1995 to December 2014.[17] We use data from Sweden, because the wide-spread use of dual class shares provides a very useful control group of stock pairs for our analysis. Dual class firms issue two types of shares, typically labeled as A- and B-shares, which provide identical ownership fractions in the underlying company and receive identical dividends, but represent different voting rights. B-shares would typically provide one vote per share, whereas the A-shares might provide 10 votes per share. Since they provide a claim to the exact same dividends, their prices are likely to be closely related.

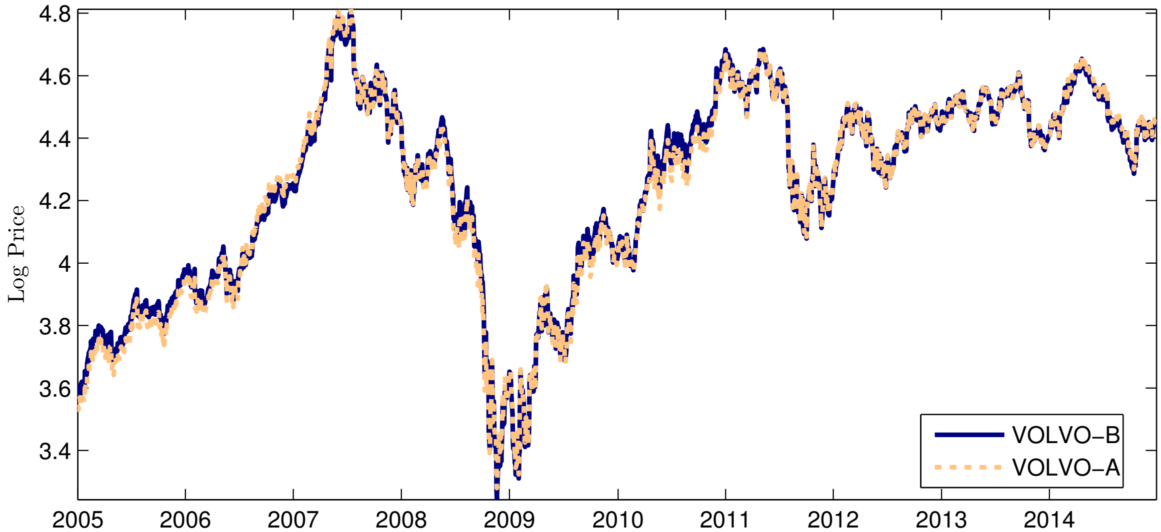
Indeed, financial theory would suggest that the A and B prices are cointegrated (as in Bossaerts (1988)). Also, as seen in the plot of the prices for the two classes of Volvo shares displayed in Figure 2, the A-B prices track each other almost perfectly. This visual evidence is certainly very compelling and suggestive of cointegration. Based on the visual evidence and the theoretical motivations, we believe that the A-B prices are likely cointegrated, or at least as close to being cointegrated as one can practically find in terms of stock prices.[18] The empirical results for the A-B pairs are subsequently used as a form of calibration for

[17] The data are from the FinBas database, which contains end-of-day stock prices, adjusted for corporate actions (e.g., stock splits and buybacks), from the Nordic Stock Exchanges. It is administered by the Swedish House of Finance. Further details are available at `https://data.houseoffinance.se/finbas/finbasInfo`.

[18] As stressed in Section IV.B, it is empirically very difficult to irrefutably verify cointegration, but it is certainly hard to imagine stronger co-movement than that illustrated in Figure 2. If one studies a graph of the *difference* between the two prices, some persistent patterns are visible, but the overall scale of these price differences is clearly small, as immediately evidenced by Figure 2. As a result, however, formal cointegration tests, which do not take into account the scale of the deviations viz a viz the original price series, can often *not* reject the null of no cointegration for the A-B pairs, despite the fact that the price co-movement for all A-B pairs in our sample look very similar to that illustrated in Figure 2. Nevertheless, we regard the A-B pairs as a relevant control group against which to compare the degree of co-movement in the ordinary pairs.

Figure 2: Log Price Series of Volvo A- and B-Shares

The figure shows the (log-) price series for the two share classes of Volvo, one of the biggest Swedish companies, in the last ten years of our sample (Jan 2005 to Dec 2014).



our empirical methods. Specifically, the A-B pairs provide a benchmark for how closely the empirical results are expected to align with the theoretical predictions, if we are indeed observing cointegrated price pairs.[19]

Our empirical strategy is thus to investigate both ordinary pairs and A-B pairs, where by "ordinary" we refer to pairs where the two stocks are issued by two different companies. The empirical analysis consists of two parts. First, we calculate empirical Sharpe ratios and other trade characteristics from the implementation of a pairs trading strategy similar to that analyzed in GGR. Second, we quantify how close the implicit restrictions of cointegration (discussed in Section II.C) are to being satisfied empirically. Both exercises provide direct evidence on how well the empirical results for the ordinary pairs approximate the theoretical model predictions, both in terms of pairs-trading returns as well as in terms of actual coefficient restrictions implied by cointegration. In addition, the results for the ordinary pairs are compared to the results for the A-B pairs. If the empirical trade characteristics

[19] To the extent that the A-B pairs are not perfectly cointegrated (see Footnote 18), the comparison to A-B pairs is conservative in the sense that the results for other non-cointegrated pairs are less likely to be clearly separated from the A-B pairs. That is, one would be less likely to find evidence *against* cointegration of other pairs, compared to the case when the A-B pairs are perfectly cointegrated.

and coefficient estimates of the A-B pairs are (much) more in line with the theoretical predictions of the cointegrated model than those of ordinary pairs, we view it as evidence that cointegration between prices of stocks issued by different companies is not likely.

Sweden is number one in the world in terms of the use of dual class shares (La Porta, Lopez-de Silanes, Shleifer and Vishny (1998)). Some of the biggest and most well known Swedish companies have dual class shares, where both classes are publicly traded on the same exchange. Moreover, the voting premium on the high voting A-shares is also among the lowest in the world, and it is lower than in the U.S. (Nenova (2003)). Before 1993, Swedish firms, apart from having A- and B-shares, could also have restricted and unrestricted share classes. As a consequence, many firms had four different types of shares. Restricted and unrestricted shares not only differed in voting rights, but also represented different cash flow rights. Moreover, only unrestricted shares could be held by foreigners. In January 1993 the distinction between restricted and unrestricted share classes was abolished, leaving firms with only A- and B-shares; for a detailed analysis of the effects of this change, see Holmén (2011). To avoid complications due to the differences between restricted and unrestricted shares, we begin our sample in January 1995, so that the market had enough time to incorporate the change in 1993. The end of the sample period is December 2014, which gives 20 years of daily data.

Table 4 provides a brief description of the sample as of June 1, 2015. There are 72 listed companies with a total market capitalization of 5403bn SEK. Most of these firms have dual class shares. However, in many cases, only one of the share classes is listed on the exchange. We need to observe the price series of both classes for our analysis, and hence we need firms with both A- and B-shares listed. There are 21 such companies representing 2271bn SEK, which is 42% of the total market capitalization. We restrict the analysis to firms for which we observe all end-of-day prices (closing price or the average of the end-of-day bid and ask prices if the closing price does not exist) during our 20-year sample period. There are 24 such companies representing 55% of the total market capitalization (2957bn SEK). Out of these

Table 4: Sample Coverage

The table describes the large cap firms on the Nasdaq Stockholm exchange as of June 1, 2015. The upper panel presents the number of firms and total market capitalization for different subsets. The lower panel presents the top 10 (in terms of market capitalization) firms.

	# Firms	Mkt. Cap (bn. SEK)
Large cap firms	72	5403
Dual class firms with both classes listed	21	2271
Firms *in the sample*	24	2975
Dual class firms *in the sample*	8	1638
	A-B listed	Mkt. Cap (bn. SEK)
Hennes & Mauritz AB	No	486
Nordea Bank AB	No	437
Ericsson, Telefonab. L M	Yes	298
Atlas Copco AB	Yes	284
Investor AB	Yes	244
Svenska Handelsbanken	Yes	242
Skandinaviska Enskilda Banken	Yes	241
Volvo AB	Yes	232
Swedbank AB	No	227
TeliaSonera AB	No	217

24 firms, there are 8 with both A- and B- shares listed, representing 30% of the total market capitalization (1898bn SEK). Altogether, there are 8 A-B pairs and 488 ordinary pairs in our sample. The lower panel of Table 4 lists the top 10 Swedish firms in terms of market capitalization. Out of the top 10 firms, six have both A- and B-shares listed on the exchange.

A. Empirical Sharpe Ratios

We start by calculating Sharpe ratios and other trade characteristics of pairs trading in our sample. The evaluation period runs from January 1996 to December 2014, since the first 12 months of the original 20 year sample has to be reserved for the first formation period, as explained below.

GGR consider the following implementation of pairs trading. During a 12-month formation period, all possible stock pairs are formed, and for each pair the sum of squared deviations (SSD) in the standardized daily price series of the two constituent stocks is calculated.[20] From all possible pairs, those with the lowest SSD are chosen for trading. During the trading

[20]The standardized price series is the cumulative total return index scaled to start at 1 SEK at the beginning of the period. The scaling is done at the beginning of both the formation and the trading periods.

period, defined as the 6-month period immediately following the 12-month formation period, a long-short position in a pair is opened whenever the standardized prices of the constituent stocks diverge by more than two standard deviations of the historical price difference observed over the formation period. In GGR's implementation, the position is closed when the two price series converge, or at the end of the trading period if convergence never occurs. Pairs that complete a round-trip are then available for trading again for that period. A typical trading portfolio is an equally weighted combination of the top 5 or top 20 pairs with the lowest SSD. GGR repeat this 12-6 implementation cycle every month, effectively mimicking a hedge fund of separate managers whose implementation cycles are staggered by one month. The monthly return on the strategy is the equally weighted monthly return across the 6 managers who are active in the given month. When calculating average returns and Sharpe ratios for this strategy, we follow the "committed-capital" approach of GGR and assume that the fund has committed capital to each of the chosen pairs, and when this capital is not invested in an open pair, it earns zero returns.

To be in line with our theoretical analysis, we modify GGR's strategy at one point: instead of holding an open position in a pair until the two price series converge, the position is held for a fixed holding period (i.e., a fixed number of trading days). Otherwise, our strategy is exactly the same as the one described above. In particular, while pairs are not held until convergence, after a pair has closed it is not eligible for trading again until it has converged; if it converged before closing, it is available for trading immediately upon closing.

Sharpe ratios and other characteristics of the pairs-trading strategies are reported in Table 5. To put the results into perspective, note that the OMXS30 index, which is the capitalization-weighted index of the 30 largest stocks of the Stockholm stock exchange, has an annualized Sharpe ratio of 0.37 over our sample period. We separately report results for trading ordinary and A-B pairs. The first three columns of Table 5 report the performance of strategies where the top 5, 8, and 20 *ordinary pairs* are traded. The last two columns report the performance of strategies where the top 5 and top 8 *A-B pairs* are traded. Note

Table 5: Empirical Characteristics of Pairs-Trading Strategies

The table presents Sharpe ratios and other characteristics of pairs-trading strategies where each position is held open for fixed number of trading days (5, 10, and 15 days in Panels A, B, and C, respectively). The first three columns correspond to the strategies where the top 5, 8, and 20 ordinary pairs (in terms of lowest SSD) are traded. The last two columns correspond to the strategies where the top 5 and top 8 A-B pairs (in terms of lowest SSD) are traded.

	Ordinary pairs			A-B pairs	
Number of traded pairs	5	8	20	5	8
A. 5-day holding period					
Annualized Sharpe ratio	1.32	1.60	1.89	2.87	2.91
Annualized Sharpe ratio (10bp one-way cost)	0.85	1.06	1.26	2.15	2.28
Annualized Sharpe ratio (30bp one-way cost)	-0.12	-0.08	-0.12	-0.27	0.04
Average trigger (%)	9.73	10.11	11.14	1.96	2.34
Trades per share in 6 months	1.36	1.33	1.21	2.21	1.81
Mean of per trade returns (%)	1.08	1.14	1.13	1.05	1.13
Std of per trade returns (%)	4.46	4.51	4.68	1.25	1.32
Annualized per trade Sharpe ratio	1.71	1.78	1.72	5.94	6.04
B. 10-day holding period					
Annualized Sharpe ratio	1.37	1.57	1.86	2.84	2.94
Annualized Sharpe ratio (10bp one-way cost)	1.00	1.16	1.35	2.07	2.27
Annualized Sharpe ratio (30bp one-way cost)	0.24	0.30	0.27	-0.24	0.11
Average trigger (%)	9.73	10.11	11.13	1.99	2.35
Trades per share in 6 months	1.32	1.29	1.17	1.99	1.64
Mean of per trade returns (%)	1.28	1.34	1.34	1.07	1.16
Std of per trade returns (%)	5.36	5.63	5.96	1.29	1.44
Annualized per trade Sharpe ratio	1.19	1.19	1.12	4.14	4.05
C. 15-day holding period					
Annualized Sharpe ratio	0.99	1.30	1.41	2.66	2.67
Annualized Sharpe ratio (10bp one-way cost)	0.68	0.96	1.01	1.89	2.00
Annualized Sharpe ratio (30bp one-way cost)	0.05	0.24	0.18	-0.20	0.13
Average trigger (%)	9.75	10.12	11.13	1.96	2.36
Trades per share in 6 months	1.28	1.26	1.14	1.78	1.49
Mean of per trade returns (%)	1.17	1.39	1.30	1.07	1.18
Std of per trade returns (%)	6.36	6.40	6.73	1.36	1.57
Annualized per trade Sharpe ratio	0.75	0.88	0.79	3.20	3.07

that in the latter case, there is no pair selection going on, since the full set of 8 A-B pairs are available for trade all the time. Different panels of Table 5 correspond to the cases when the open positions are held for 5, 10, and 15 trading days, respectively.

The first row in each panel reports the before-cost Sharpe ratios for the committed-capital strategy. When trading ordinary pairs, all Sharpe ratios fall in the range between 1 and 1.9, which is in line with previous studies. GGR report only marginally higher Sharpe ratios when studying a different market (the U.S.) and a different time period (1962-2002). Do and Faff (2010), who study the same strategies as GGR using U.S. data, report similar Sharpe ratios to ours for the subperiods that overlap with our sample period. When trading A-B pairs, the

Sharpe ratios fall in the range between 2.6 and 3.0. That is, before-cost Sharpe ratios from trading A-B pairs are considerably higher than what can be attained when trading ordinary pairs.

We also consider the effect of transaction costs in Table 5. At least three types of costs emerge when implementing a pairs trading strategy: commissions, short selling fees, and the implicit cost of market impact (Do and Faff (2012)). The effect of commissions and market impact needs to be considered whenever a position is initiated or closed in a given stock. Since one complete pairs trade involves two round-trips, the associated transaction cost will be four times the per-stock one-way cost (commission plus market impact). Do and Faff (2012), who study the transaction costs associated with pairs trading in the U.S., estimate the average one-way cost to be around 30 bps in the period 1989-2009. Do and Faff (2012) also take into account short-selling fees by including a constant loan fee of 1% per annum payable over the life of a given pairs trade. In our empirical implementation we ignore the short-selling fees, since they are negligible compared to the other costs according to the estimates of Do and Faff (2012), and deduct two times the one-way cost both at the initiation and at the close of each pairs trade to calculate the after-cost return corresponding to each strategy.[21]

The second and third row in each panel of Table 5 report the after-cost Sharpe ratios for the committed-capital strategies. Specifically, we consider both a low-cost case, with a one-way cost of 10 bps, as well as the 30 bps one-way cost estimated by Do and Faff (2012). With a 10 bps one-way cost, ordinary pairs earn Sharpe ratios between 0.7 and 1.4, while trading A-B pairs leads to Sharpe ratios around 2. That is, when transaction costs are low, pairs trading still performs considerably better in the case of A-B pairs. However, when the one-way cost is 30 bps, all Sharpe ratios decrease dramatically: for the ordinary pairs the

[21]Since short selling fees are assumed to be proportional to the length of the trade, they are relatively small for strategies with short holding periods. For example, the 1% annual loan fee considered by Do and Faff (2012) translates into 2 bps for the strategy that holds each position for 5 trading days. The one-way cost of 30 bps is realistic according to Do and Faff (2012), who consider typical institutional investors who trade on the U.S. market. If actual transaction costs in Sweden are higher than those in the U.S., the transaction-cost adjusted Sharpe ratios become even smaller.

Sharpe ratios are never above 0.3 and for the A-B pairs the Sharpe ratios are close to zero or negative.[22] That is, when a realistic level of transaction costs is used, the returns from pairs trading are greatly reduced or extinguished. This is particularly true for the A-B pairs.

Table 5 also shows the average price deviation that triggers a pairs trade (i.e., the average two standard deviation outcome of the price differences), as well as how often the pairs trade. For the ordinary pairs, the average trigger is about 10 percent.[23] For the A-B pairs, the average trigger is much smaller, at around two percent. On average, an ordinary pair trades around 1.3 times per 6-month period, whereas the A-B pairs trade a bit more often at around 2 times per 6 months.

There are some key differences between the Sharpe ratios reported in the first row of each panel in Table 5 and the ones we study in the theoretical section. The empirical (annualized) Sharpe ratios of Table 5 correspond to the monthly returns from trading a portfolio of pairs, where each "manager's" capital earns zero returns when not invested in a pair. On the other hand, in the theoretical analysis we study *per-trade* Sharpe ratios from trading a *single pair*. In order to be closer to our theoretical analysis and to provide more details regarding the differences between A-B pairs and ordinary pairs, we therefore also report statistics for the per-trade returns. That is, the final three rows in each panel of Table 5 report the mean and standard deviation of all per-trade returns, along with the per-trade annualized Sharpe ratios. The latter are simply defined as the ratios of the average per-trade returns over the standard deviation of the per-trade returns, scaled to an annual basis. These per-trade Sharpe ratios are thus direct empirical proxies for the theoretical Sharpe ratios derived earlier in the paper.

The per-trade return statistics further emphasize that trading A-B pairs leads to considerably

[22] Schultz and Shive (2010) analyze a form of pairs trading in U.S. dual-class shares, which should be similar to the A-B shares in our sample, and find that such strategies can yield substantial returns. In their study, they take into account bid-ask spreads but *not* other trading costs. Their results might therefore be most comparable to our low-transaction cost scenario; for instance, Jones (2002) reports average half-spreads around 10 bps on Dow Jones stocks in the year 2000, which is in the middle of the sample period (1993-2006) used by Schultz and Shive (2010).

[23] This average opening trigger is larger than the average 5 percent trigger in GGR, but quite similar to what is found by Do and Faff (2012), who consider a number of different pair formations and report an average opening spread around 8 percent across all pairs over the sample period 1989-2009.

higher Sharpe ratios than trading ordinary pairs. For each holding period, the annualized Sharpe ratios for the A-B pairs are more than three times higher than the respective Sharpe ratios for the ordinary pairs. The mean return for a given pairs trade is around 1.2%, for both ordinary- and A-B pairs, and the differences in Sharpe ratios are almost entirely driven by a much greater variance in the ordinary pairs returns. Since the full round-trip transaction cost for a pairs trade is 120 bps (when the one-way cost per stock is 30bps), this level of transaction costs effectively wipes out almost all of the returns, as seen in the after-cost Sharpe ratios for the committed-capital strategy in Table 5 (row 3 in each panel).

Overall, the properties of the A-B pairs trading returns line up reasonably well with the implications from the theoretical analysis. Pre-cost Sharpe ratios are large (upwards of 6.0 on a per-trade basis), and not dissimilar from the theoretical ones obtained under cointegration. However, given the small spreads that trigger a trade and the subsequent small returns, the after-cost Sharpe ratios are close to zero, consistent with the discussion around Corollary 1 in Section III.D.[24] In practice, there are likely arbitrageurs that perform some form of pairs trading on the A-B pairs, essentially enforcing the no-arbitrage relationship between them. If these arbitrageurs are able to trade at relatively low transaction costs, such trading might be profitable, and these profits might provide a fair return on the investments in infrastructure needed to trade at low costs.

In contrast, the per trade Sharpe ratios for the ordinary pairs are much smaller than those for the A-B pairs, while the trigger spreads for the ordinary pairs are much larger than for the A-B pairs. Pairs-trading in ordinary pairs is profitable before costs, but the after-cost Sharpe ratios for these pairs also approach zero for reasonable transaction costs. This is consistent with ordinary pairs *not* being cointegrated, since under cointegration the large opening spreads should also translate into large expected returns, which should survive

[24]The small spreads between the A-B prices suggest a very high correlation between the innovations to the two processes, such that the two processes are affected by almost identical shocks (see Section III.D). To empirically assess the correlations between innovations for the ordinary pairs and the A-B pairs, we used the innovations from VMA(5) estimations of the pairwise price processes and calculated their correlation for all the pairs in our sample. The correlations are all in the range 0.9 to 0.98 for A-B pairs, whereas they are all below 0.68 for ordinary pairs.

transactions costs quite easily (again, see the discussion in Section III.D).

B. Empirical Evidence of Cointegration

We finally turn to an empirical evaluation of pairwise cointegration of stock prices. There is a long econometric tradition of testing for cointegration among economic time series. However, the formal evaluation of long-run relationships, like cointegration, is fraught with difficulties and several decades of empirical research has produced few solid empirical "facts". Therefore, instead of attempting to provide a yes/no answer to the question of stock price cointegration using formal tests, we adopt a more gradual empirical strategy, where we try to quantify whether pairs of stocks are "close to" or "far from" being cointegrated. We make use of the implicit restrictions of cointegration discussed in Section II.C, and analyze how close these restrictions are to being satisfied in the case of ordinary stock pairs. We compare the results from ordinary pairs to the control group of A-B pairs, which, at a minimum, should be "close to" cointegrated. Similar to the previous empirical section, the sample consists of 8 A-B pairs and 488 ordinary pairs. The sample period is from January 1995 to December 2014.

The first restriction implied by cointegration, shown in equation (11), is on the long-run MA matrix, $C(1)$, and reads as

$$\psi_{12} = 1 + \psi_{22} \quad \text{and} \qquad \psi_{21} = 1 + \psi_{11} \; . \tag{26}$$

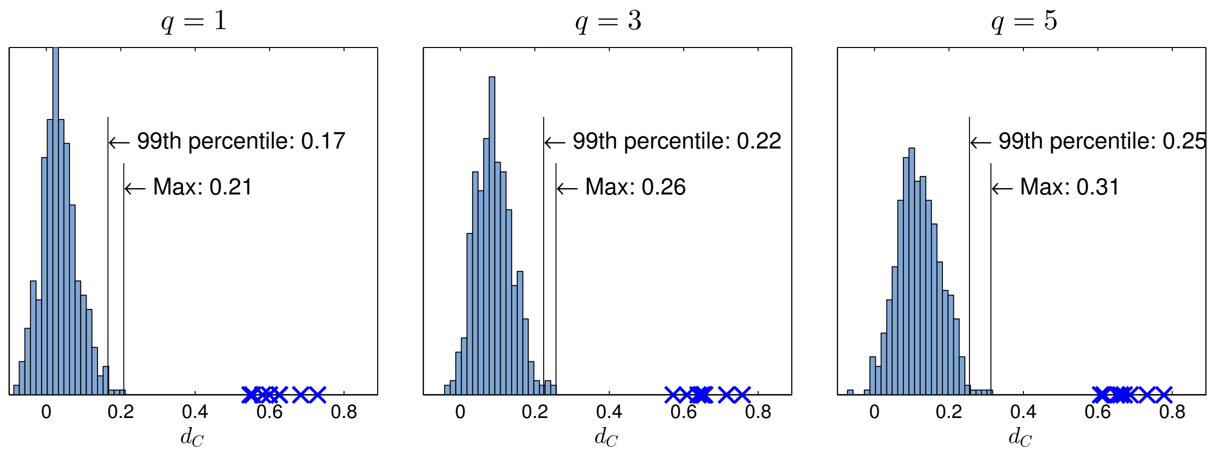
To study this restriction, we estimate VMAs of orders $q \in \{1, 3, 5\}$ for each possible pair of stocks in our sample and then calculate

$$d_C \equiv \frac{(\psi_{12} - \psi_{22}) + (\psi_{21} - \psi_{11})}{2} \; . \tag{27}$$

For a pair of cointegrated stock prices the value of d_C should be close to one. The histograms in each graph of Figure 3 describe the distribution of d_C for ordinary stock pairs. The $\times$

Figure 3: Cointegration Restrictions on the VMA Coefficients

The graphs present the distribution of d_C from equation (27) in our sample. The histograms in each graph represent the distribution of d_C for ordinary stock pairs (488 pairs). The $\times$ markers correspond to the A-B pairs (8 pairs). The separate graphs correspond to the order of the estimated VMA(q) process with $q \in \{1, 3, 5\}$.



markers represent the A-B pairs in the sample. The ordinary pairs are clearly separated from the A-B pairs in all graphs of Figure 3. In the case of ordinary stock pairs, d_C is never above 0.31, while all d_C values lie between 0.5 and 0.8 in case of the A-B pairs. The value of d_C does not actually reach 1 even for the A-B pairs, corroborating our earlier argument that the formal validation of cointegration is very hard as well as the possibility that the A-B pairs are not perfectly cointegrated. Our key empirical observation is that the coefficient estimates are much further away from satisfying the cointegrating restrictions for *all* the ordinary pairs than for *any* of the A-B pairs.

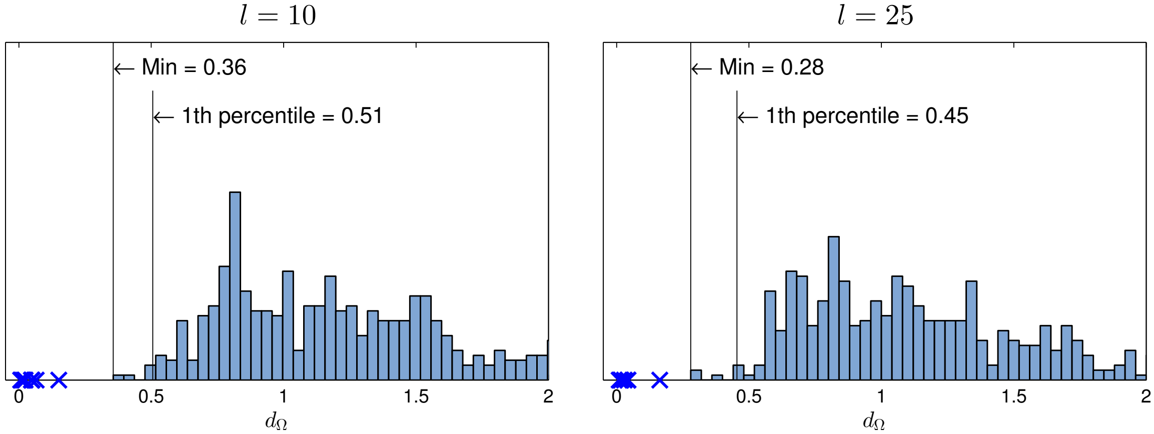
The second restriction we investigate is on the long-run covariance matrix of returns, Ω, and is shown in equation (14). Cointegration implies that all four elements of Ω are identical. In order to evaluate this restriction, we first estimate the long-run covariance matrix for all possible pairs in the sample using the Newey and West (1987) estimator with lag lengths $l \in \{10, 25\}$. The following statistic is subsequently calculated for each pair:

$$d_\Omega \equiv \left| \ln\left(\frac{\omega_{21}}{\omega_{11}}\right) \right| + \left| \ln\left(\frac{\omega_{22}}{\omega_{11}}\right) \right| \ . \tag{28}$$

The value of d_Ω is zero under cointegration and positive otherwise. The histograms in Figure 4

Figure 4: Cointegration Restrictions on the Long-Run Covariance Matrix

The graphs present the distribution of d_Ω from equation (28) in our sample. The histograms in each graph represent the distribution of d_Ω for ordinary stock pairs (488 pairs). The $\times$ markers correspond to the A-B pairs (8 pairs). The separate graphs correspond to the lag length of the estimator with $l \in \{10, 25\}$.



describe the distribution of d_Ω for ordinary stock pairs, while the $\times$ markers correspond to the A-B pairs. The ordinary pairs are clearly separated from the A-B pairs again, as d_Ω for *all* the ordinary pairs lies further away from satisfying the cointegrating restrictions than for *any* of the A-B pairs.

To be clear, Figures 3 and 4 show the values of d_C and d_Ω, respectively, for *all* ordinary pairs that can be formed in our sample of stocks. Most of these pairs would clearly not be strong candidates for pairs trading, according to any selection criterion, and clearly one should not expect *all* of these pairs to be cointegrated. Rather, the main message of Figure 3 and Figure 4 is that *none* of these ordinary pairs are anywhere near as close to satisfying the cointegration restrictions as *any* of the A-B pairs. As emphasized previously, the A-B pairs are likely close to cointegrated (see also Footnotes 18 and 19), and one can view the outcomes of the A-B pairs as a form of calibration result against which to compare the ordinary pairs. The clear separation between the ordinary pairs and the A-B pairs therefore strongly suggest that none of the ordinary pairs are cointegrated.

V. Conclusion

Cointegration of certain pairs of stock prices provides a seemingly simple and clear explanation for why pairs trading would be profitable. However, cointegration among stock prices turns out to be a very powerful concept. In a theoretical analysis, we show that cointegration among stock prices would imply annualized Sharpe ratios that could easily exceed 10, for a single pair, in a simple pairs trading strategy. This result suggest either (i) that cointegration among ordinary stocks is very unlikely, or (ii) that the transient component in stock prices is longer-lived (i.e., returns exhibit longer serial correlation) than is usually believed. In an empirical analysis, we attempt to distinguish between these two possible explanations, and find strong evidence in favour of the first conjecture. Namely, that stock prices are very rarely cointegrated.

In conclusion, our study suggests that cointegration is probably not the right starting point when trying to understand pairs trading. Instead, one needs to consider weaker concepts, and/or accept that pairs trading opportunities are likely of a transient nature, which do not fit well into a model of a permanent long-run relationship as stipulated by cointegration.

A. Simulation Procedure

The steps of the simulation are as follows.

1. We start simulating values of Δy_t using the model in equation (1) and, at the same time, also create $y_t = \Delta y_t + y_{t-1}$ and $\beta' y_t$ values. For starting values we use $\epsilon_t = 0$ for all $t \leq 0$ and $y_0 = 1$. The innovations, ϵ_t, are drawn from an *iid* normal distribution. For a given $h(j)$ function, the c_j coefficients are specified according to equation (A35), with $a = d = \frac{0.1}{H_\infty}$. This parametrization keeps the long-run VMA matrix, $C(1)$, identical across different specifications of $h(j)$. As seen in Theorem 2, the distribution of the Sharpe ratios is, in fact, invariant to the values of a and d. This result is also confirmed for the simulated Sharpe ratios in simulations not reported in the paper.

2. After simulating the first 10^5 periods, we estimate the unconditional standard deviation of the difference between the two price processes, i.e., $\sigma_{\beta' y} = \sqrt{Var(\beta' y_t)}$.

3. We continue to simulate Δy_t, y_t, and $\beta' y_t$ values for 10^6 more periods, and use these realizations to estimate the number of initiated trades. A pairs trade is first initiated when $|\beta' y_t| > 2\sigma_{\beta' y}$. If the pair converges during p periods (the holding-period), the pair is eligible to trade again immediately after the p periods. If the pair does not converge during p periods, it becomes eligible for trading only after it has converged. Once the pair is eligible for trading, the next pairs trade is initiated when $|\beta' y_t| > 2\sigma_{\beta' y}$ again. We count the number of initiated trades over the 10^6 simulated periods and scale the result so that it corresponds to a 6-month interval (125 trading periods).

4. We continue simulating Δy_t, y_t, and $\beta' y_t$ values until the condition $r\sigma_{\beta' y} - \varepsilon < \beta' y_\tau < r\sigma_{\beta' y} + \varepsilon$ is met at a given time period τ. This is when one particular pairs trade is initiated. We use $r \in \{-1, -2\}$, corresponding to one- and two standard deviation strategies, respectively, and we set $\varepsilon = 5 \cdot 10^{-4}$. We record the p-period return after the pairs trade is initiated, defined as $r_{\tau \to \tau+p} = \sum_{j=1}^{p} \Delta \beta' y_{\tau+j}$ for $p = 1, ..., 250$. We

also record the number of periods needed until the two price processes converge, i.e., we find the first τ' such that $\beta' y_{\tau'} > 0$ and $\tau' > \tau$, and record $\tau' - \tau$.

5. We repeat step 4. until we have observations from $N = 10^6$ initiated pairs trades. The p-period returns from these N different simulated pairs trades enable us to calculate the $E_t[r_{t\to t+p}]$, $Var_t(r_{t\to t+p})$, and $SR_t(r_{t\to t+p})$ values. The recorded time to convergence values enable us to produce the plots with the percentage of converged trades.

In Appendix B we also provide simulation results in a VAR setting. The procedure is the same as the one described above, with the only difference that the Δy_t values are simulated from the VAR model in equation (A2).

B. Pairs Trading Returns in a VAR Setting

We consider the VAR(1) case, and focus on one-period pairs trading returns. Suppose the bivariate prices, $y_t = (y_{1,t}, y_{2,t})'$, are generated according to a cointegrated VAR(1) process,

$$y_t = \Pi_1 y_{t-1} + \epsilon_t. \tag{A1}$$

The vector error correction model (VECM) format is given by

$$\Delta y_t = \Pi y_{t-1} + \epsilon_t, \tag{A2}$$

with $\Pi = \Pi_1 - I$. By standard results, if y_t is cointegrated with vector β, one can write $\Pi = \alpha\beta'$, for some vector $\alpha = (\alpha_1, \alpha_2)$, such that

$$\beta' y_t = (\beta' + \beta'\alpha\beta')\, y_{t-1} + \beta'\epsilon_t = (1 + \beta'\alpha)\, \beta' y_{t-1} + \beta'\epsilon_t \tag{A3}$$

and

$$(A4) \qquad \Delta y_t = \alpha\beta' y_{t-1} + \epsilon_t.$$

For $\beta = (1, -1)$,

$$(A5) \qquad \beta'\alpha = \alpha_1 - \alpha_2 = -(\alpha_2 - \alpha_1) \equiv -\gamma.$$

Let $w_t \equiv \beta' y_t = y_{1,t} - y_{2,t}$ be the difference between the two (cointegrated) price processes, and define $\xi_t \equiv \beta'\epsilon_t$. The price difference w_t then follows an AR(1) process,

$$(A6) \qquad w_t = (1-\gamma) w_{t-1} + \xi_t.$$

Under cointegration, this process must be stationary, which holds for $|1-\gamma| < 1$, or equivalently, $\gamma \in (0, 2)$.

The returns on the one-period pairs trading strategy is given by

$$(A7) \qquad r_{t\to t+1} \equiv \Delta\beta' y_{t+1} = \beta'\alpha\beta' y_t + \beta'\epsilon_{t+1}.$$

The conditional expectation and conditional variance at time t are given by

$$(A8) \qquad \begin{aligned} E_t\left[r_{t\to t+1}\right] &= E_t\left[\beta'\alpha\beta' y_t + \beta'\epsilon_{t+1}\right] = \beta'\alpha\beta' y_t = -\gamma\left(\beta' y_t\right) \\ Var_t\left(r_{t\to t+1}\right) &= Var\left(\beta'\epsilon_{t+1}\right) = \beta' Var\left(\epsilon_{t+1}\right)\beta = \beta'\Sigma\beta. \end{aligned}$$

The unconditional variance of $\beta' y_t$ is given by

$$(A9) \qquad \begin{aligned} Var\left(\beta' y_t\right) &= Var\left(\left(1+\beta'\alpha\right)\beta' y_{t-1}\right) + Var\left(\beta'\epsilon_t\right) \\ &= (1-\gamma)^2 Var\left(\beta' y_{t-1}\right) + \beta'\Sigma\beta \end{aligned}$$

Table A1: Properties of Pairs-Trading in the VAR(1) Model.

γ	0.008	0.01	0.02	0.05	0.10	0.20	0.50	0.99
SR_{ann}	2.00	2.24	3.18	5.06	7.25	10.54	18.26	31.31

By stationarity of the cointegrating relationship, $Var(\beta' y_t) = Var(\beta' y_{t-1})$, and thus

$$Var(\beta' y_t) = \frac{\beta'\Sigma\beta}{1-(1-\gamma)^2} = \frac{\beta'\Sigma\beta}{\gamma(2-\gamma)}. \tag{A10}$$

The Sharpe ratio for the pairs trading strategy is then equal to

$$SR_{t,t\to t+1} = \frac{E_t[r_{t\to t+1}]}{\sqrt{Var_t(r_{t\to t+1})}} = -\frac{\gamma(\beta' y_t)}{\sqrt{\beta'\Sigma\beta}}. \tag{A11}$$

If one sets the price deviation, $\beta' y_t$, equal to two standard deviations,

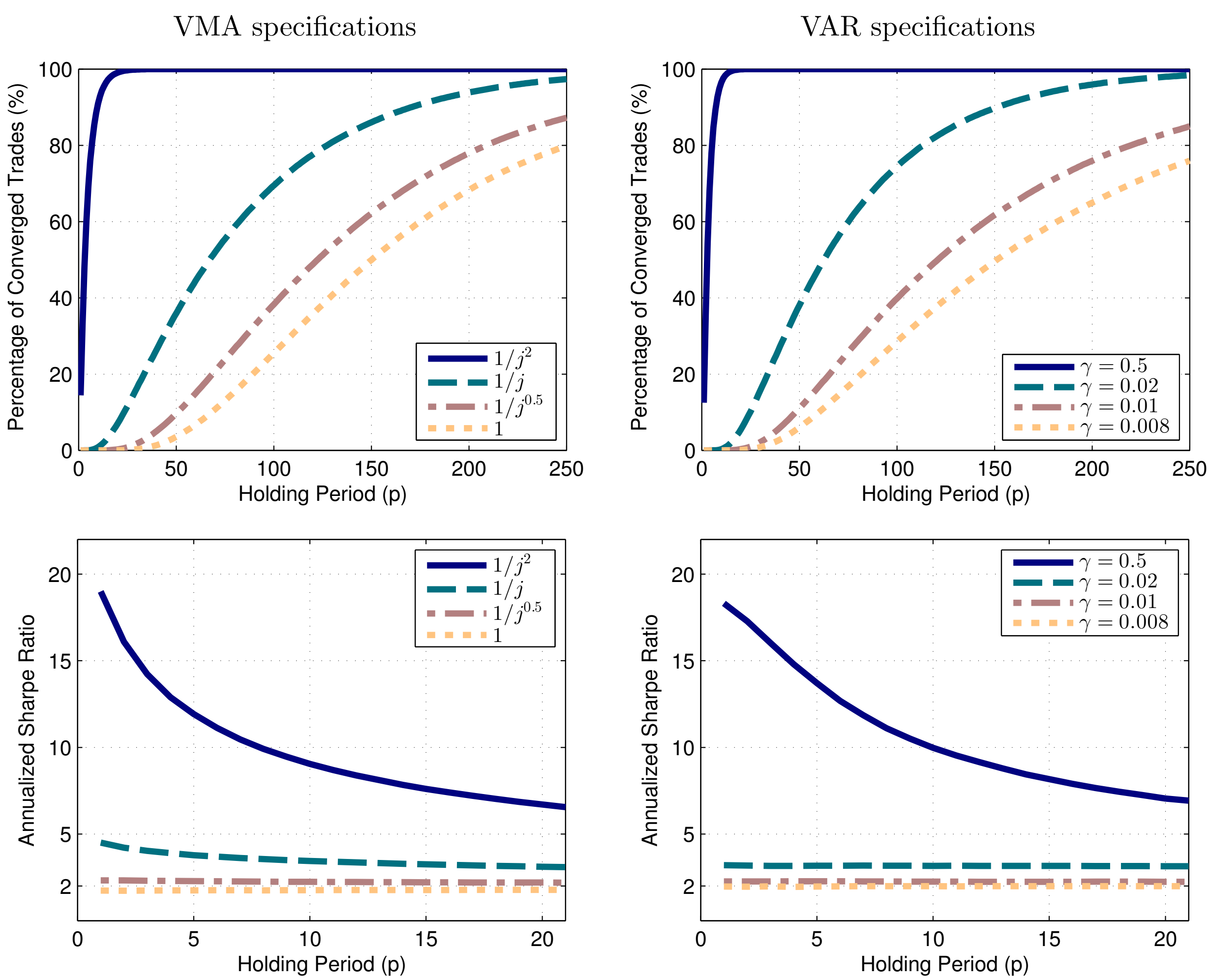
$$-\beta' y_t = 2\sqrt{Var(\beta' y_t)} = 2\sqrt{\frac{\beta'\Sigma\beta}{\gamma(2-\gamma)}}, \tag{A12}$$

it follows that

$$SR_{t,t\to t+1} = -\frac{\gamma(\beta' y_t)}{\sqrt{\beta'\Sigma\beta}} = 2\frac{\gamma\sqrt{\frac{\beta'\Sigma\beta}{\gamma(2-\gamma)}}}{\sqrt{\beta'\Sigma\beta}} = 2\sqrt{\frac{\gamma}{(2-\gamma)}}. \tag{A13}$$

In the VAR framework outlined above, the difference between the two cointegrated price processes follow an AR(1) process with the AR-parameter given by $1-\gamma$. For γ close to zero, the difference between the two prices is almost non-stationary, and their convergence towards each other is slow. For γ close to one, there is little persistence in the difference between the prices, and convergence happens quickly. The parameter γ thus plays a similar role here, to the role of q in the VMA setting. Table A1 shows the annualized Sharpe ratios, across different values of γ, for a two standard deviation strategy with a one-day holding period. The results in the table highlight that the Sharpe ratio of the pairs trading strategy quickly becomes very large as γ drifts away from zero.

Figure A1: Properties of a Two Standard Deviation Strategy for VMA and VAR Processes



In order to more directly compare the VMA and VAR formulations, Figure A1 shows the convergence properties and Sharpe ratios of a two standard deviation pairs trading strategy under VMA specifications (graphs on the left) and VAR specifications (graphs on the right). The results in the figure are based on simulations. For the VMA specifications, we pick $q = 250$ and $h(j) = \{1, 1/j^{0.5}, 1/j, 1/j^2\}$; that is, we reproduce the exact same results as in the right column of Figure 1. For the VAR specifications, the $\gamma = \{0.008, 0.01, 0.02, 0.5\}$ values are chosen such that the convergence properties of the trading strategy for each value is similar to the convergence properties of one of the VMA specifications. As shown in the top graphs of Figure A1, the percentage of converged trades shows a similar pattern for specific pairs of VMA and VAR specifications. The corresponding pairs trading Sharpe ratios are shown in the two bottom graphs, and the main take-away of Figure A1 is that VMA and VAR specifications with similar convergence patterns also have similar Sharpe ratios.

C. Proofs and Additional Theorems

Proof of Theorem 1. Under the presence of cointegration, the q-period pairs trading returns are equal to

(A14) $$r_{t\to t+q} \equiv \sum_{k=1}^{q} \Delta\beta' y_{t+k} = -\beta' \sum_{k=1}^{q} \Delta\tilde{\epsilon}_{t+k} = -\beta' \left(\tilde{\epsilon}_{t+q} - \tilde{\epsilon}_t\right).$$

In the general $VMA(\infty)$ case, $\tilde{\epsilon}_t = \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t-j}$, and

$$\text{(A15)} \qquad \begin{aligned} \tilde{\epsilon}_{t+q} - \tilde{\epsilon}_t &= \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t+q-j} - \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t-j} \\ &= \left(\tilde{c}_0 \epsilon_{t+q} + \tilde{c}_1 \epsilon_{t+q-1} + ... + \tilde{c}_{q-1} \epsilon_{t+1} + \tilde{c}_q \epsilon_t + \tilde{c}_{q+1} \epsilon_{t-1} + ...\right) \\ &\quad - \left(\tilde{c}_0 \epsilon_t + \tilde{c}_1 \epsilon_{t-1} + \tilde{c}_2 \epsilon_{t-2} + \tilde{c}_3 \epsilon_{t-3} + ...\right) \\ &= \tilde{c}_0 \epsilon_{t+q} + \tilde{c}_1 \epsilon_{t+q-1} + ... + \tilde{c}_{q-1} \epsilon_{t+1} \\ &\quad + \left(\tilde{c}_q - \tilde{c}_0\right) \epsilon_t + \left(\tilde{c}_{q+1} - \tilde{c}_1\right) \epsilon_{t-1} + \left(\tilde{c}_{q+2} - \tilde{c}_2\right) \epsilon_{t-2} + ... \\ &= \sum_{k=0}^{q-1} \tilde{c}_k \epsilon_{t+q-k} + \sum_{j=0}^{\infty} \left(\tilde{c}_{j+q} - \tilde{c}_j\right) \epsilon_{t-j}. \end{aligned}$$

Further, $\tilde{c}_j = \sum_{s=j+1}^{\infty} c_s$, which gives,

$$\text{(A16)} \qquad \tilde{c}_{j+q} - \tilde{c}_j = \sum_{s=j+1+q}^{\infty} c_s - \sum_{s=j+1}^{\infty} c_s = -\left(c_{j+1} + c_{j+2} + ... + c_{j+q}\right) = -\sum_{k=1}^{q} c_{j+k},$$

and

$$\text{(A17)} \qquad \tilde{\epsilon}_{t+q} - \tilde{\epsilon}_t = \sum_{k=0}^{q-1} \tilde{c}_k \epsilon_{t+q-k} - \sum_{j=0}^{\infty} \left(\sum_{k=1}^{q} c_{j+k} \right) \epsilon_{t-j}.$$

That is,

$$\text{(A18)} \qquad r_{t \to t+q} = -\beta' \left(\tilde{\epsilon}_{t+q} - \tilde{\epsilon}_t\right) = -\beta' \left(\sum_{k=0}^{q-1} \tilde{c}_k \epsilon_{t+q-k} - \sum_{j=0}^{\infty} \left(\sum_{k=1}^{q} c_{j+k} \right) \epsilon_{t-j} \right).$$

The conditional variance at time t of $r_{t \to t+q}$ is thus equal to,

(A19)

$$Var_t\left(r_{t \to t+q}\right) = Var_t\left(-\beta' \left(\sum_{k=0}^{q-1} \tilde{c}_k \epsilon_{t+q-k} \right) \right) = \beta' \sum_{k=0}^{q-1} Var\left(\tilde{c}_k \epsilon_{t+q-k}\right) \beta = \sum_{k=0}^{q-1} \left(\beta' \tilde{c}_k \Sigma \tilde{c}_k' \beta\right).$$

Similarly, the conditional expectation at time t is given by

$$\text{(A20)} \qquad \begin{aligned} E_t\left[r_{t\to t+q}\right] &= E_t\left[-\beta'\left(\sum_{k=0}^{q-1}\tilde{c}_k\epsilon_{t+q-k} - \sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}\right)\right] \\ &= \beta'\left(\sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}\right). \end{aligned}$$

Note that (with initial conditions set to zero),

$$\text{(A21)} \qquad \beta' y_t = -\beta'\tilde{\epsilon}_t = -\beta'\sum_{j=0}^{\infty}\tilde{c}_j\epsilon_{t-j} = -\beta'\sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty}c_s\epsilon_{t-j}.$$

Suppose now that we have a finite $VMA(q)$ model. In that case, $c_j = 0$ for $j > q$. Thus,

$$\text{(A22)} \qquad \begin{aligned} \beta' y_t &= -\beta'\sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty}c_s\epsilon_{t-j} = -\beta'\sum_{j=0}^{q-1}\sum_{s=j+1}^{q}c_s\epsilon_{t-j} \\ &= -\beta'\sum_{j=0}^{q-1}\sum_{k=1}^{q-j}c_{j+k}\epsilon_{t-j} = -E_t\left[r_{t\to t+q}\right], \end{aligned}$$

since

$$\text{(A23)} \qquad E_t\left[r_{t\to t+q}\right] = \beta'\left(\sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}\right) = \beta'\left(\sum_{j=0}^{q-1}\sum_{k=1}^{q-j}c_{j+k}\epsilon_{t-j}\right).$$

The unconditional variance of $\beta' y_t$, in the $VMA(q)$ case, is given by

$$\text{(A24)} \qquad Var\left(\beta' y_t\right) = Var\left(\beta'\tilde{\epsilon}_t\right) = \beta' Var\left(\sum_{j=0}^{q-1}\tilde{c}_j\epsilon_{t-j}\right)\beta = \sum_{j=0}^{q-1}\beta'\tilde{c}_j\Sigma\tilde{c}_j'\beta = Var_t\left(r_{t\to t+q}\right).$$

■

Theorem A1 *Suppose* $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ *is a* $k\times 1$ *dimensional returns process, with* $\epsilon_t \equiv iid(0,\Sigma)$, $C(L) = \sum_{j=0}^{\infty}c_jL^j$, *and* $C(1) \neq 0$. *The corresponding price process is given by* y_t, *and the (one-period) returns on the pairs trading strategy is defined as* $r_{t\to t+1} \equiv \Delta\beta' y_{t+1}$. *If* y_t *is cointegrated with cointegration vector* β, *the following results hold.*

i. *The time t conditional expected one-period return is given by*

$$\text{(A25)} \qquad E_t\left[r_{t\to t+1}\right] = -\beta' y_t - \beta' \sum_{j=0}^{\infty} \tilde{c}_{j+1}\epsilon_{t-j} = \beta' \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}.$$

ii. *The time t conditional variance of the one-period return is given by*

$$\text{(A26)} \qquad Var_t\left(r_{t\to t+1}\right) = \beta'\Sigma\beta.$$

iii. *The time t conditional Sharpe ratio for the one-period return is given by*

$$\text{(A27)} \quad SR_{t\to t+1} = \frac{E_t\left[r_{t\to t+1}\right]}{\sqrt{Var_t\left(r_{t\to t+1}\right)}} = \frac{-\beta' y_t - \beta' \sum_{j=0}^{\infty} \tilde{c}_{j+1}\epsilon_{t-j}}{\sqrt{\beta'\Sigma\beta}} = \frac{\beta' \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}}{\sqrt{\beta'\Sigma\beta}}.$$

Proof of Theorem A1. From the proof of Theorem 1,

$$\text{(A28)} \qquad r_{t\to t+1} = -\beta'\left(\tilde{\epsilon}_{t+1} - \tilde{\epsilon}_t\right) = -\beta'\left(\tilde{c}_0\epsilon_{t+1} - \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}\right),$$

where the last equality uses equation (A17). The conditional variance at time t is then equal to,

$$\text{(A29)} \qquad Var_t\left(r_{t\to t+1}\right) = Var_t\left(-\beta'\left(\tilde{c}_0\epsilon_{t+1} - \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}\right)\right) = Var_t\left(\beta'\tilde{c}_0\epsilon_{t+1}\right),$$

Further, $\tilde{c}_0 = \sum_{s=1}^{\infty} c_s = C(1) - I$ and $\beta' C(1) = 0$, resulting in

$$\text{(A30)} \qquad Var_t\left(r_{t\to t+1}\right) = Var_t\left(\beta'\left(C(1) - I\right)\epsilon_{t+1}\right) = \beta' Var\left(\epsilon_{t+1}\right)\beta = \beta'\Sigma\beta.$$

The conditional expectation at time t is given by

$$\text{(A31)} \qquad E_t\left[r_{t\to t+1}\right] = E_t\left[-\beta'\left(\tilde{c}_0\epsilon_{t+1} - \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}\right)\right] = \beta' \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}.$$

Alternatively, by equation (8) with initial conditions equal to zero, $\beta' y_t = -\beta' \tilde{\epsilon}_t$, and

$$\text{(A32)} \qquad \begin{aligned} E_t \left[r_{t \to t+1} \right] &= E_t \left[-\beta' \left(\tilde{\epsilon}_{t+1} - \tilde{\epsilon}_t \right) \right] = \beta' \tilde{\epsilon}_t - E_t \left[\beta' \tilde{\epsilon}_{t+1} \right] \\ &= -\beta' y_t - E_t \left[\beta' \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t-j+1} \right] = -\beta' y_t - \beta' \sum_{j=0}^{\infty} \tilde{c}_{j+1} \epsilon_{t-j}. \end{aligned}$$

■

Proof of Theorem 2. Note that

(A33)

$$C(1) = \sum_{j=0}^{\infty} c_j = I + \sum_{j=0}^{\infty} h(j) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv I + H_\infty \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 + aH_\infty & bH_\infty \\ cH_\infty & 1 + dH_\infty \end{bmatrix}.$$

Under cointegration with vector $\beta = (1, -1)'$, $C(1)$ must satisfy the restrictions in (11), which implies that the following parameter restrictions apply,

$$\text{(A34)} \qquad b = \frac{1}{H_\infty} + d, \text{ and } c = \frac{1}{H_\infty} + a.$$

Thus, without loss of generality, we can write

$$\text{(A35)} \qquad c_j = h(j) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = h(j) \begin{bmatrix} a & \frac{1}{H_\infty} + d \\ \frac{1}{H_\infty} + a & d \end{bmatrix}.$$

By Theorem A1,

$$\text{(A36)} \qquad Var_t \left(r_{t \to t+1} \right) = \beta' \Sigma \beta = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sigma_{11} + \sigma_{22} - 2\sigma_{12}.$$

Using equation (A31) in the proof of Theorem A1, the Sharpe ratio can then be written as

$$\text{(A37)} \qquad SR_{t \to t+1} = \frac{\beta' \sum_{j=0}^{\infty} c_{j+1} \epsilon_{t-j}}{\sqrt{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}}.$$

The denominator in this Sharpe ratio is a constant. This reflects the fact that the time t variance of the pairs trading returns is constant across time and does not depend on the current state of the price processes. Plugging in for the above parametrization of c_j and $\beta = (1, -1)'$,

$$\text{(A38)} \qquad SR_{t \to t+1} = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} a & \frac{1}{H_\infty} + d \\ \frac{1}{H_\infty} + a & d \end{bmatrix} \sum_{j=0}^{\infty} h(j+1)\, \epsilon_{t-j}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}} = \frac{\begin{bmatrix} -\frac{1}{H_\infty} & \frac{1}{H_\infty} \end{bmatrix} \sum_{j=1}^{\infty} h(j)\, \epsilon_{t-j+1}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}$$

If ϵ_t is iid $N(0, \Sigma)$, it follows that $\sum_{j=1}^{\infty} h(j)\, \epsilon_{t-j+1}$ is also normally distributed with mean zero and variance

$$\text{(A39)} \qquad Var\left(\sum_{j=1}^{\infty} h(j)\, \epsilon_{t-j+1} \right) = \sum_{j=1}^{\infty} h(j)^2\, \Sigma \equiv \Sigma H_\infty^{(2)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} H_\infty^{(2)},$$

with $H_\infty^{(2)} \equiv \sum_{j=1}^{\infty} h(j)^2$. The numerator in the Sharpe ratio in equation (A38) is thus normally distributed with mean zero and variance,

$$\text{(A40)}\, Var\left(\begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix}' \sum_{j=1}^{\infty} h(j)\, \epsilon_{t-j+1} \right) = \begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix}' \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} H_\infty^{(2)} \begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix} = \frac{H_\infty^{(2)}}{H_\infty^2} (\sigma_{11} + \sigma_{22} - 2\sigma_{12}).$$

The Sharpe Ratio is then distributed as

$$\text{(A41)} \qquad SR_{t \to t+1} \sim \frac{N\left(0, \frac{H_\infty^{(2)}}{H_\infty^2} (\sigma_{11} + \sigma_{22} - 2\sigma_{12})\right)}{\sqrt{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}} = N\left(0, \frac{H_\infty^{(2)}}{H_\infty^2}\right).$$

■

Proof of Corollary 1. Using equation (8), and the definition of $\tilde{c}_j$ in equation (6),

$$\text{(A42)}\quad Var\left(\beta' y_t\right) = Var\left(\beta'\tilde{\epsilon}_t\right) = \left(\sum_{j=0}^{\infty} \beta'\tilde{c}_j \Sigma \tilde{c}_j' \beta\right) = \left(\sum_{j=0}^{\infty} \left(\sum_{s=j+1}^{\infty} \beta' c_s\right) \Sigma \left(\sum_{s=j+1}^{\infty} c_s'\beta\right)\right).$$

By the proof of Theorem 2, it must hold that $c_j = h\left(j\right) \begin{bmatrix} a & \frac{1}{H_\infty} + d \\ \frac{1}{H_\infty} + a & d \end{bmatrix}$. For $\beta = (1, -1)$, it follows that $\beta' c_s = h\left(s\right) \begin{bmatrix} -\frac{1}{H_\infty} & \frac{1}{H_\infty} \end{bmatrix}$ and

$$\begin{aligned}
\text{(A43)}\quad & Var\left(\beta' y_t\right) \\
&= \begin{bmatrix} -\frac{1}{H_\infty} & \frac{1}{H_\infty} \end{bmatrix} \left(\sum_{j=0}^{\infty} \left[\left(\sum_{s=j+1}^{\infty} h\left(s\right)\right) \Sigma \left(\sum_{s=j+1}^{\infty} h\left(s\right)\right)\right]\right) \begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{H_\infty} & \frac{1}{H_\infty} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix} \left(\sum_{j=0}^{\infty} \left[\left(\sum_{s=j+1}^{\infty} h\left(s\right)\right) \left(\sum_{s=j+1}^{\infty} h\left(s\right)\right)\right]\right) \\
&= \frac{1}{H_\infty^2} \left(\sigma_{11} + \sigma_{22} - 2\sigma_{12}\right) \sum_{j=0}^{\infty} \left[\left(\sum_{s=j+1}^{\infty} h\left(s\right)\right)^2\right] \\
&= \left(\sigma_{11} + \sigma_{22} - 2\sigma_{12}\right) \sum_{j=0}^{\infty} \left[\frac{\left(\sum_{s=j+1}^{\infty} h\left(s\right)\right)^2}{\left(\sum_{s=1}^{\infty} h\left(s\right)\right)^2}\right],
\end{aligned}$$

where the last equality follows from the definition $H_\infty \equiv \sum_{j=1}^{\infty} h\left(j\right)$. ■

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